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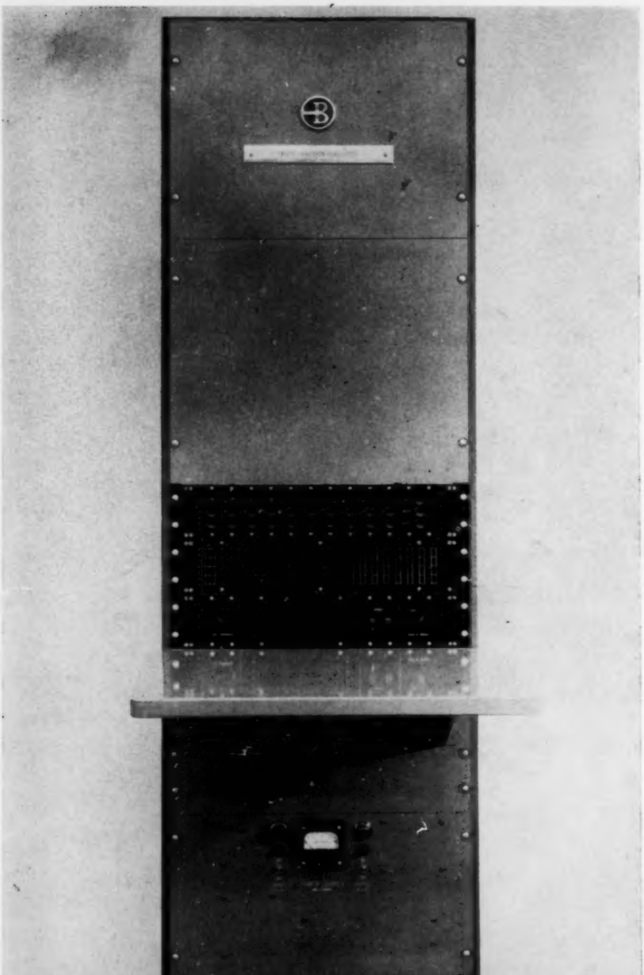

BURROUGHS TRUTH FUNCTION EVALUATOR

An Analysis of a Logical Machine Using Parenthesis-Free Notation

1. Introduction. ŁUKASIEWICZ' parenthesis-free notation[1] permits a simple and easily mechanizable process of truth-table computation. We shall describe this process and prove some relevant theorems. A 10-variable relay machine employing this method for a two-valued logic has been constructed and used by the Burroughs Corporation (see frontispiece) and it is feasible in the present state of the computer art to construct a 25-variable machine which would reduce all evaluants of a 250-character formula in a two-valued logic with monadic and dyadic operators in about an hour. The operation of such a machine will be described in terms of the manipulations which it performs upon a formal propositional language.

2. The Language. We consider any language L based on a quadruple $\langle C, \boldsymbol{W}, P, \boldsymbol{F}\rangle$ whose terms will be characterized in this section[2]. C is a finite non-empty set of *characters*.

(*Example 1:* The theory will be illustrated from time to time by the use of a specific sample L whose characters (members of C) are the following: the functors N (negation), K (dyadic conjunction), A (dyadic alternation); the propositional variables, p, q, r; the truth-constants 1 (truth), 0 (falsity).)

Definition 1: Any finite sequence of characters (including the null sequence) is a *formula*.

'Λ' will designate the null formula. Except for 'Λ' and 'π' (to be introduced later), all lower case and upper case Greek letters (with and without natural number subscripts) will range over all characters and all formulas respectively. Juxtaposition in the syntax language will be used to denote juxtaposition in L. We will use the following terminology pertaining to formulas.

Definition 2:

A (length): $\boldsymbol{L}(\Delta)$ is the number of character tokens in Δ.

Let $\Delta = \Phi\Psi$, then

B (tail): $\boldsymbol{T}_i(\Delta) = \Psi$ where $i \leqq \boldsymbol{L}(\Delta)$ and $\boldsymbol{L}(\Psi) = i$, or $i > \boldsymbol{L}(\Delta)$ and $\Psi = \Delta$;

C (head): $\boldsymbol{H}_i(\Delta) = \Phi$ where $i \leqq \boldsymbol{L}(\Delta)$ and $\boldsymbol{L}(\Phi) = i$, or $i > \boldsymbol{L}(\Delta)$ and $\Phi = \Delta$.

$\boldsymbol{W}$ is a function assigning to each character (element of C) an integral *weight* $\leqq 1$.

(*Example 2:* N has weight 0, K and A have weight -1, the variables and truth-constants have weight 1.)

The following concepts are defined in terms of weight.

Definition 3:

A (character degree): $\boldsymbol{D}(\delta) = 1 - \boldsymbol{W}(\delta)$.

B (formula weight): $\boldsymbol{W}(\Delta)$ is the sum of the weights of the character tokens of Δ; $\boldsymbol{W}(\Lambda) = 0$.

C (max. weight, tail): $\boldsymbol{W}_{\text{Max}}(\Delta) = \text{Max}(\boldsymbol{W}[\boldsymbol{T}_i(\Delta)])$ for $i > 0$.

D (min. weight, tail): $W_{\text{Min}}(\Delta) = \text{Min}(W[T_i(\Delta)])$ for $i > 0$.

E (positive formula): Δ is a positive formula if and only if $W_{\text{Min}}(\Delta) > 0$.

F (well-formed formula): Δ is a well-formed formula if and only if Δ is a positive formula and $W(\Delta) = 1$.

(*Example 3:* $pKqN$ is not a positive formula; $ApqNr$ is positive but not well-formed; $KpNq$ is well-formed.)

Note that the non-recursive definition 3F shows that a single scan of a formula from right to left is sufficient to determine whether it is well-formed.

P is a non-empty subset of C whose elements are *truth-constants*. 'π' with or without natural number subscripts will range over this set. P must satisfy the condition

(1) $$W(\pi) = 1.$$

F is a function which assigns to each character of degree > 0 a "truth function"; i.e., which satisfies the conditions

(2a) $$\text{If } D(\delta) > 0, \quad \text{then } F(\delta\pi_{D(\delta)}\cdots\pi_1)\epsilon P;$$

(2b) $$F(\pi) = \pi.$$

(*Example 4:* In the sample language P consists of 0 and 1; the function F applied to K gives $F(K00) = 0$, $F(K01) = 0$, $F(K10) = 0$, $F(K11) = 1$.)

We can now define the ordinary concept of (propositional) variable.

Definition 4: δ is a *variable* if and only if $W(\delta) = 1$ and δ is not in P.

An important relation between positive formulas and well-formed formulas which we will need is given by:

THEOREM I: (A) *Δ is a positive formula if and only if Δ can be partitioned (i.e., divided into a sequence of disjoint segments which exhaust it) into exactly $W(\Delta) \geqq 1$ well-formed formulas.* (B) *There is at most one partition of a positive formula Δ into well-formed formulas.*[3]

Proof: (A) The "if" part is obvious. The "only if" part is proved as follows. Let Γ be the shortest head of positive weight of a positive formula Δ. Hence, for any $i > 0$, $W[T_i(\Gamma)] > 0$; and since Γ is not null it is a positive formula. Again, since Γ is the shortest head of Δ of positive weight, $W[H_a(\Gamma)] \leqq 0$ (where $a = L(\Gamma) - 1$); and since Γ is a positive formula, $W[T_1(\Gamma)] = 1$; hence $W(\Gamma) \leqq 1$. But then $W(\Gamma) = 1$ and Γ is a well-formed formula. It follows that $W[T_b(\Delta)] = W(\Delta) - 1$, for $b = L(\Delta) - L(\Gamma)$; then if $T_b(\Delta)$ is not null it is a positive formula and this process may be repeated until Δ is partitioned into $W(\Delta)$ well-formed formulas.

(B) The proof is by induction on $L(\Delta)$. If $L(\Delta) = 1$, Δ is a single character and "B" obviously holds. Suppose "B" holds for all positive formulas of length $< n$; consider a positive formula Δ with $L(\Delta) = n$. If there are two partitions of Δ, one will have a first (leftmost) well-formed formula Φ of length $\leqq$ the length of the first well-formed formula $\Phi\Psi$ in the other partition. But $W(\Phi\Psi) - W(\Phi) = W(\Psi) = 0$ since $W(\Phi\Psi) = W(\Phi) = 1$. Therefore $\Psi = \Lambda$, else $W_{\text{Min}}(\Phi\Psi) = 0$ implying $\Phi\Psi$ is not well-formed. If $\Phi = \Phi\Psi = \Delta$, then "B" has been established for Δ. Otherwise $\Delta = \Phi\Gamma = \Phi\Psi\Gamma$ where Γ is a positive formula and $L(\Gamma) < L(\Delta)$. In this case the inductive hypothesis shows that "B" holds for Γ and hence for $\Phi\Gamma = \Delta$.

Theorem I makes possible the following definition.

Definition 5 (partition function): If $\Delta = \Delta_J \cdots \Delta_j \cdots \Delta_1$ and each Δ_j is well-formed, then $P_j(\Delta) = \Delta_j$.

(*Example 5:* For $\Delta = pKpqNr$, $P_1(\Delta) = p$, $P_2(\Delta) = Kpq$, $P_3(\Delta) = Nr$.)

With the aid of Theorem I and Definition 3A it can be shown that our formulation of the language L is easily reducible to a more conventional formulation. For example, Δ is a well-formed formula if and only if it is of the form $\delta\Delta_{D(\delta)} \cdots \Delta_1$, where each Δ_d is well-formed. Note that $D(\delta)$ is the number of well-formed formulas following δ in the above decomposition; if $D(\delta) > 0$, δ is an operator of the propositional calculus and $D(\delta)$ is its degree in the ordinary sense, while if $D(\delta) = 0$, δ is well-formed by itself and hence is a truth-constant or propositional variable.

An interesting property peculiar to the parenthesis-free notation is given by:

THEOREM II: *Every formula of a language L is a segment of some well-formed formula of L if and only if L contains at least one character of negative weight.*

Proof: The proof of the "if" part is as follows. Let ω be a character of negative weight. For Δ an arbitrary formula of the language L, $\Psi\Delta\Phi$ is well-formed where Φ consists of $1 - W_{\mathrm{Min}}(\Delta)$ occurrences of π and Ψ consists of a formula of weight -1 (e.g., ω followed by $-1 - W(\omega)$ occurrences of π) repeated $W(\Delta\Phi) - 1$ times. The proof of the "only if" part is as follows. $W(\pi\pi) = 2$, and, since $\pi\pi$ is a segment of some well-formed formula Δ, L must contain a character of negative weight in order that $W(\Delta) = 1$.

The need for a machine to do truth-table computation occurs only for languages containing at least one symbol of negative weight and a multiplicity of symbols of weight 1.

3. The Machine. With the aid of the two following definitions we can describe, for any given language L, the construction of a machine to evaluate formulas of L.

Definition 6 (the set of specification functions): G is a specification function if and only if (1) if δ is a variable, $G(\delta) \epsilon P$ and (2) if δ is not a variable, $G(\delta) = \delta$.

(Hereafter 'S' will represent an arbitrary specification function.)

Definition 7 (evaluation function): $E_S(\Lambda) = \Lambda$; if $\delta\Delta$ is a positive formula, $E_S(\delta\Delta) = F(S(\delta)H_{D(\delta)}[E_S(\Delta)])T_a[E_S(\Delta)]$ where $a = L[E_S(\Delta)] - D(\delta)$.

(*Example 6:* For the formula of Example 5, $\Delta = pKpqNr$, the recursive evaluation is as follows: let the specification, S, of the variables be $p = 1$, $q = 0$, $r = 0$, then $E_S[T_1(\Delta)] = F[S(r)] = 0$, $E_S[T_2(\Delta)] = F(N0) = 1$, $\cdots$, $E_S[T_5(\Delta)] = F(K10)1 = 01$, $E_S(\Delta) = F(p)01 = 101$.)

The following lemma shows that $L(H_{D(\delta)}[E_S(\Delta)]) = D(\delta)$ and hence that E_S is defined for all positive formulas.

LEMMA: If Δ is a positive formula, then $E_S(\Delta)$ is of the form $\pi_n\pi_{n-1}\cdots\pi_1$ where $n = W(\Delta)$.

Proof: That all characters are π's follows directly from the conditions (2a and 2b) which characterize F. That $n = W(\Delta)$ is established by an induction on $L(\Delta)$.

Note that if Δ is well-formed, $L[E_S(\Delta)] = 1$; it is easy to show that when Δ is well-formed, $E_S(\Delta)$ is its truth-value relative to S in the ordinary sense.

A number of well-formed formulas may be evaluated concurrently by juxtaposing them to form (by Theorem I) a positive formula Δ. Let $\Delta = \delta_{L(\Delta)} \cdots \delta_i \cdots \delta_1$, let m be the number of truth-constants in the language L (i.e., L is an m-valued logic), and let v be the number of distinct variables of Δ. The machine makes m^v scans of Δ, one for each distinct specification of the variables, producing each time $E_S(\Delta)$; a single complete scan for a given S with the accompanying computation determines the "Sth" *major cycle.* Each major cycle is divided into $L(\Delta)$ *minor cycles*, the ith minor cycle encompassing the processing of the ith character δ_i.

The machine consists of two basic components, a Memory and an Evaluator. The *Memory*[4] (e.g., a magnetic drum, an acoustic delay line) stores Δ and during each major cycle sends it characters $\delta_{L(\Delta)} \cdots \delta_1$ to the Evaluator in order of ascending subscripts. During the Sth major cycle the *Evaluator* realizes the recursive function $E_S(\Delta)$ by producing successively the $E_S[T_i(\Delta)]$. It does this by means of three parts: a Specifier, a Function Switch, and a Register.

The *Specifier* (e.g., an electronic counter with switching gates) effects the sequence of S functions required by Δ; during the ith minor cycle it receives δ_i and produces $S(\delta_i)$, and at the end of the Sth major cycle it introduces a new S. During the ith minor cycle of the Sth major cycle the *Function Switch* (e.g., an array of electronic switching gates) receives $S(\delta_i)$ from the Specifier, $H_{D(\delta_i)}(E_S[T_{i-1}(\Delta)])$ from the Register, and produces the new character $F[S(\delta_i)H_{D(\delta_i)}(E_S[T_{i-1}(\Delta)])]$. This new character is sent to the *Register* (e.g., an electronic shifting register) which by shifting an amount $W(\delta_i)$ (a positive value indicates a right shift, a negative value a left shift) substitutes it for $H_{D(\delta_i)}(E_S[T_{i-1}(\Delta)])$.

4. Some Theorems. The following theorem justifies the use of a positive formula Δ to compute the truth-values of its well-formed components $P_j(\Delta)$.

THEOREM III: *If Δ is a positive formula then* $E_S(\Delta) = E_S[P_{W(\Delta)}(\Delta)] \cdots E_S[P_1(\Delta)]$.

(*Example 7:* In the formula of Examples 5 and 6, $E_S(\Delta) = E_S(p)E_S(Kpq)E_S(Nr)$ which, for the S of Example 6, evaluates to 101.)

Proof: The theorem follows directly from the fact that, if Φ and Ψ are positive formulas or null, $E_S(\Phi\Psi) = E_S(\Phi)E_S(\Psi)$. This may be proved by an induction on $L(\Phi)$. It is obviously true for $\Phi = \Lambda$. We assume it to be true for some Γ and show that it holds for $\gamma\Gamma$. By Definition 7 $E_S(\gamma\Gamma\Psi) = F(S(\gamma)H_{D(\gamma)}[E_S(\Gamma\Psi)])\,(T_a[E_S(\Gamma\Psi)])$, where $a = L[E_S(\Gamma\Psi)] - D(\gamma)$, and by the inductive hypothesis $E_S(\gamma\Gamma\Psi) = F(S(\gamma)H_{D(\gamma)}[E_S(\Gamma)E_S(\Psi)])(T_b[E_S(\Gamma)E_S(\Psi)])$ where $b = L[E_S(\Gamma)E_S(\Psi)] - D(\gamma)$. Since $\gamma\Gamma$ is a positive formula, $L[E_S(\Gamma)] \geqq D(\gamma)$ and we have $E_S(\gamma\Gamma\Psi) = F(S(\gamma)H_{D(\gamma)}[E_S(\Gamma)])H_c[E_S(\Gamma)]E_S(\Psi)$, where $c = L[E_S(\Gamma)] - D(\gamma)$; hence $E_S(\gamma\Gamma\Psi) = E_S(\gamma\Gamma)E_S(\Psi)$.

For a given positive formula Δ the Register must be able to store a (positive) formula of length $\operatorname*{Max}_{i=1}^{L(\Delta)} [L(E_S[T_i(\Delta)])]$. The following two theorems show how this quantity depends upon the structure of Δ.

THEOREM IV: *If Δ is a positive formula then* $\operatorname*{Max}_{i=1}^{L(\Delta)} [L(E_S[T_i(\Delta)])] = W_{\mathrm{Max}}(\Delta)$.

(*Example 8:* In Example 6 it can be seen that $\operatorname*{Max}_{i=1}^{L(\Delta)=6} [L(E_S[T_i(\Delta)])]$

$= L[E_S(\Delta)] = 3 = W_{\text{Max}}(\Delta)$. It may be checked that the evaluants of $T_3(\Delta)$ and $T_4(\Delta)$ which were not given will not change this maximum.)

Proof: By the Lemma and Definition 3C.

THEOREM V: *If Δ is a positive formula, then* $W_{\text{Max}}(\Delta) = \operatorname*{Max}_{j=1}^{W(\Delta)} (W_{\text{Max}}[P_j(\Delta)] + j - 1)$.

Proof: Consider all the tails of Δ and note that each complete $P_j(\Delta)$ in a tail contributes a weight of 1.

This theorem shows that, given a set of well-formed formulas to be evaluated as a positive formula or to be used as arguments for a commutative operator, a formula of smallest W_{Max} may be formed by placing these well-formed formulas in order of ascending W_{Max} from left to right.

In designing a specific machine to carry out our process some decisions must be made concerning the relative sizes of the Memory and the Register. Our concluding theorem gives information relevant to this decision.

THEOREM VI: *Let $M_{W,D}$ be the set of well-formed formulas Ψ such that* (1) *the maximum degree of any character of Ψ is $D > 1$ and* (2) $W_{\text{Max}}(\Psi) = W$. *Then* (A) *for any $\Psi \in M_{W,D}$ there is a $\Phi \in M_{W,D}$ such that $L(\Phi) \geqq L(\Psi)$ and* (B) $L(\Psi) \geqq W + \left\{\frac{W-1}{D-1}\right\}$, *where $\{z\}$ is the smallest integer $\geqq z$.*

Proof: For part (A) consider for any $\Psi \in M_{W,D}$ the formula $\Psi_1 = \delta\pi_D \cdots \pi_2\Psi$, where $D(\delta) = D$. Clearly Ψ_1 satisfies (1). By Theorem V, $W_{\text{Max}}(\Psi_1) = \text{Max}(D,W)$ which equals W since, for any positive formula Δ, $W_{\text{Max}}(\Delta) \geqq$ the maximum degree of any character of Δ. Hence $\Psi_1 \in M_{W,D}$ and $L(\Psi_1) > L(\Psi)$. To establish (B) consider any $\Psi \in M_{W,D}$. By (2) Ψ must contain at least W characters of weight 1. Since $W(\Psi) = 1$ the sum of the weights of its characters of negative weight is at most $1 - W$; by (1) no character of Ψ has a weight less than $1 - D$; since $D > 1$, Ψ must contain at least $\left\{\frac{W-1}{D-1}\right\}$ characters of negative weight. Hence $L(\Psi) \geqq W + \left\{\frac{W-1}{D-1}\right\}$.

For a language L containing characters of all degrees between 2 and D inclusive, if $M_{W,D}$ is non-empty, it contains a well-formed formula of the minimum length which consists of $\left\{\frac{W-1}{D-1}\right\}$ characters of negative weight followed by W characters of weight 1.

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[1] JAN ŁUKASIEWICZ, *Aristotle's Syllogistic from the standpoint of Modern Formal Logic.* Oxford, 1951, p. 78.

[2] In this section we employ the results of STANISŁAW JAŚKOWSKI, KARL MENGER, KARL SCHRÖTER, and D. C. GERNETH. See PAUL ROSENBLOOM, *The Elements of Mathematical Logic.* New York, 1950, ch. 4, sec. 1, for a discussion and further references.

[3] Theorem I implies that under the operation juxtaposition the set of positive formulas together with the two-sided identity Λ is a semigroup which is generated by the set of well-formed formulas.

[4] For a description of computer components see ENGINEERING RESEARCH ASSOCIATES, *High Speed Computing Devices.* New York, 1950.

Polynomial-Like Approximation

Frequently, in machine computations, one may substitute for a function of relatively complicated analytic structure a function of simpler form without appreciable error. Moreover, if the function is given tabularly, a considerable reduction of tabular input data (load on the memory) may be achieved. In what follows, we deal with real-valued functions of continuous real variables $x, y, \cdots$. Computationally, it is usually better to work with variables assuming finitely many discrete values. In the formulas which follow, however, the discrete analogues are apparent, integration to be replaced by summation, etc. We have in mind multivariate approximations by polynomial-like forms, in particular, by functions M of the form

$$M(x, y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y), \tag{1}$$

in the bivariate case. Some of the following results are generalizable to functions of more than two variables.

Of special interest is the so-called slide rule form, $f(x) + g(y)$. We shall give formulas for approximating to a given real-valued continuous function, defined over the unit square $0 \leq x, y \leq 1$ to minimize, respectively, each of the following measures of goodness of fit:

$$\text{(a)} \qquad \int_0^1 \int_0^1 [z(x, y) - f(x) - g(y)]^2 dxdy, \quad \text{and}$$

$$\text{(b)} \qquad \max_{0 \leq x,\, y \leq 1} |z(x, y) - f(x) - g(y)|.$$

A simple variational technique yields for (a):

$$f(x) = \int_0^1 z(x, y)dy + c_0, \quad \text{and}$$

$$g(y) = \int_0^1 z(x, y)dx + c_1, \quad \text{where}$$

c_0 and c_1 are any real numbers such that

$$c_0 + c_1 = -\int_0^1 \int_0^1 z(x, y)dxdy.$$

Recursive formulas for (b) are based heuristically on the fact that given two numbers a and b, one can approximate to them by a single number c; taking c to be the arithmetic mean of a and b minimizes the maximum absolute error. The recursive formulas for (b) are as follows:

$$f_{n+1}(x) = \tfrac{1}{2} \{ \max_y [z(x, y) - g_n(y)] + \min_y [z(x, y) - g_n(y)]\},$$

$$g_n(y) = \tfrac{1}{2} \{ \max_x [z(x, y) - f_n(x)] + \min_x [z(x, y) - f_n(x)]\},$$

and f_0 arbitrary but continuous, $n = 0, 1, \cdots$. DILIBERTO & STRAUS[1] have shown quite generally that the above process converges to a pair of continu-

ous functions (f, g) which has the required property. Included in their discussion is a precise method, discovered independently by the writer, for estimating the value of the minimum. In this connection, the reader is referred to their paper.

We now return to the more general form (1). Let $n > 0$ be a prescribed integer. We assume that z (continuous over the unit square) is not of the form (1) for this particular n. We shall give two sets of formulas for approximating to z by continuous functions of the form (1), which satisfy respectively:

(c) The error vanishes on some rectangular grid of lines $x = x_1, x_2, \cdots, x_n$, $y = y_1, y_2, \cdots, y_n$ ($x_i \neq x_j$ and $y_i \neq y_j$ if $i \neq j$)

(d) $$\int_0^1 \int_0^1 \left[z(x, y) - \sum_{j=1}^{n} \phi_j(x)\psi_j(y) \right]^2 dxdy \text{ is minimized.}$$

For (c), functions ϕ_j, ψ_j can be computed successively as follows. Choose a point (x_1, y_1) such that $z(x_1, y_1) \neq 0$ and two constants c_1, c_1' such that $c_1c_1' = 1/z(x_1, y_1)$ and set

$$\phi_1(x) = c_1 z(x, y_1) \quad \text{and} \quad \psi_1(y) = c_1' z(x_1, y).$$

We compute ϕ_2 and ψ_2 in precisely the same manner, except that z is now replaced by a new function z_2 given by

$$z_2(x, y) = z(x, y) - \phi_1(x)\psi_1(y).$$

Thus, we choose a point (x_2, y_2) such that $z_2(x_2, y_2) \neq 0$, and two constants c_2, c_2' such that $c_2c_2' = 1/z_2(x_2, y_2)$ and set

$$\phi_2(x) = c_2 z_2(x, y_2) \quad \text{and}$$
$$\psi_2(y) = c_2' z_2(x_2, y).$$

We continue in this manner, with

$$z_{k+1}(x, y) = z_k(x, y) - \phi_k(x)\psi_k(y)$$

until we have the required number of functions. A simple inductive argument shows that condition (c) is satisfied.

That the solution of problem (d), i.e. finding a least squares fit to z of the form (1), bears a direct relationship to the theory of integral operators in Hilbert space was pointed out by G. W. Brown a few years ago. His observations are essentially these:

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the *largest* n eigenvalues of the symmetric kernel

$$K(x, y) = \int_0^1 z(x, t)z(y, t)dt$$

and $\phi_1, \phi_2, \cdots, \phi_n$ be corresponding eigenfunctions (normalized so that $\int_0^1 \phi_j^2(x)dx = 1$) and set

$$\psi_j(y) = \int_0^1 z(x, y)\phi_j(x)dx, \quad j = 1, 2, \cdots, n,$$

then the ϕ's and ψ's defined in this way form a solution to (d), the ϕ's being mutually orthogonal and the ψ's being mutually orthogonal. The computations can be carried on sequentially. Let λ_1 be the largest eigenvalue of K, ϕ_1 a corresponding normalized eigenfunction and ψ_1 defined as above. Having found ϕ_1 and ψ_1 we form the symmetric kernel, K_1 associated with the residual function $z(x, y) - \phi_1(x)\psi_1(y)$. Then, the largest eigenvalue of K_1 is the next largest eigenvalue of K and we take ϕ_2 to be the corresponding normalized eigenfunction with ψ_2 defined as above. We continue in this manner until we have the required number of functions. Incidentally, for $n = 1$, a variational technique yields the necessary conditions for an extremum:

$$\phi(x) = \int_0^1 z(x, y)\psi(y)dy / \int_0^1 \psi^2(y)dy,$$

$$\psi(y) = \int_0^1 z(x, y)\phi(x)dx / \int_0^1 \phi^2(x)dx.$$

These can be used to generate an iterative computation for ϕ and ψ. However, questions of convergence, proper initiating functions to achieve the largest eigenvalue, etc. seem to be difficult. One final remark—the minimum value of $\int_0^1 \int_0^1 [z(x, y) - \sum_{j=1}^{n} \phi_j(x)\psi_j(y)]^2 dxdy$ is precisely the sum of the remaining eigenvalues of K, i.e. the sum of the eigenvalues minus the sum of the largest n eigenvalues, the former sum being equal to $\int\int z^2 dxdy$.

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[1] S. P. Diliberto & E. G. Straus, "On the approximation of a function of several variables by the sum of functions of fewer variables," *Pacific Jn. of Math.*, v. 1, p. 195–210, 1951.

A Logarithm Algorithm

The method of calculating logarithms given in this paper is quite unlike anything previously known to the author and seems worth recording because of its mathematical beauty and its adaptability to high-speed computing machines. Although there are well known methods[1] which involve continued fractions, these methods invariably utilize the *analytic* properties of the logarithm *function* and not the *arithmetic* properties of the *individual* logarithm. The first version of this algorithm is based directly upon such arithmetic continued fractions. In a subsequent skeletonized modification, however, continued fractions no longer appear explicitly.

Let $a_0 > a_1 > 1$ be given. To find $\log_{a_0} a_1$ we determine the two sequences

$$a_2, a_3, \cdots$$
$$n_1, n_2, \cdots,$$

where the n's are positive integers, by the relations

$$(1) \qquad \begin{aligned} a_i^{n_i} &< a_{i-1} < a_i^{n_i+1} \\ a_{i+1} &= a_{i-1}/a_i^{n_i}. \end{aligned}$$

We define the complete quotient $x_1 > 1$ by

$$a_0 = a_1^{n_1+1/x_1}.$$

Since

$$a_1 = a_2^{x_1}$$

we write

$$x_1 = n_2 + 1/x_2$$

and so on. Thus we have the continued fraction

$$\log_{a_0} a_1 = \frac{1}{n_1 +}\,\frac{1}{n_2 +}\,\frac{1}{n_3 +}\cdots.$$

For example let $a_0 = 10$, $a_1 = 2$. Then

$$2^3 < 10 < 2^4.$$

Therefore $n_1 = 3$ and $a_2 = 10/2^3 = 1.25$. Further:

$$1.25^3 < 2 < 1.25^4.$$

Thus $n_2 = 3$ and $a_3 = 2/1.25^3 = 1.024$. Continuing we obtain

i	n_i	a_i
0	—	10
1	3	2
2	3	1.25
3	9	1.024
4	2	1.009741958
5	2	1.004336279
6	4	1.001041545

It follows that

$$\log_{10} 2 = \frac{1}{3 +}\,\frac{1}{3 +}\,\frac{1}{9 +}\,\frac{1}{2 +}\,\frac{1}{2 +}\,\frac{1}{4 +}\cdots.$$

Breaking off the continued fraction at successive terms we find:

i	rational approximation to $\log_{10} 2$	
1	1/3	= .33333333
2	3/10	= .30000000
3	28/93	= .30107527
4	59/196	= .30102041
5	146/485	= .30103093
6	643/2136	= .30102996.

Since $\log_{10} 2 = .30102999566$ we see that each successive approximation gives us approximately one more correct decimal place than the previous one. This rate of convergence, one decimal place per cycle, is typical. If one chooses at random an a_0 and an a_1, one will "almost always" find such a rate of convergence. A method of estimating this universal rate is given below.

Both phases of the recurrence rules given in (1) may be carried out simultaneously. One divides a_{i-1} by a_i; then the quotient by a_i; then that quotient

by a_i, etc., until the resulting quotient is less than a_i. Then this last quotient is a_{i+1} and the number of divisions is n_i. In fact, this part of the process involves *only division*.

The second part of the process, namely, that of obtaining the successive fractions, may similarly be reduced to *only addition*, but at first we note that from the theory of continued fractions if P_{i-1}/Q_{i-1} and P_i/Q_i are the rational approximations which include the terms up to $1/n_{i-1}$ and $1/n_i$ respectively; then

$$\frac{P_{i+1}}{Q_{i+1}} = \frac{P_{i-1} + n_{i+1}P_i}{Q_{i-1} + n_{i+1}Q_i}. \tag{2}$$

For example, above we had

$$P_2/Q_2 = 3/10,\ P_3/Q_3 = 28/93,\ n_4 = 2;$$

and therefore

$$P_4/Q_4 = 59/196.$$

However, for an automatic computing machine we recommend the following variation which abstracts the contents of the last two paragraphs. We maintain six registers A, B, C, D, E, and F. At each inning we do one of two things:

Operation I (if $A \geq B$):
We put A/B in A, $C + E$ in C, and $D + F$ in D.

Operation II (if $A < B$):
We interchange A and B, C and E, D and F. We start with a_0 in A, a_1 in B, 1 in C and F, and 0 in D and E. The latest approximation to $\log_{a_0}a_1$ is always E/F.

In the example above we would obtain:

	A	B	C	D	E	F
Op. I	10	2	1	0	0	1
	5	2	1	1	0	1
	2.5	2	1	2	0	1
	1.25	2	1	3	0	1
Op. II →						
Op. I	2	1.25	0	1	1	3
	1.6	1.25	1	4	1	3
	1.28	1.25	2	7	1	3
	1.024	1.25	3	10	1	3
Op. II →						
	1.25	1.024	1	3	3	10

It is readily seen that in this variation we need *not* assume $a_0 > a_1$, as was done in (1), but merely that $a_0 \geq 1$ and $a_1 \geq 1$. Further, if one or both of these numbers are less than 1, then since

$$\log_a x^{-1} = -\log_a x, \quad \log_{1/a} x = -\log_a x, \quad \log_{1/a} x^{-1} = \log_a x,$$

we may proceed as before after taking the reciprocals of those numbers less than 1. We then multiply the fraction obtained by $(-1)^m$ where m is the number of reciprocals taken.

If it happens that the logarithm is a rational number, for instance $\log_8 4 = 2/3$, then at some point B becomes 1, the exact log is obtained and no further changes in E or F occurs. For example:

A	B	C	D	E	F
8	4	1	0	0	1
2	4	1	1	0	1
4	2	0	1	1	1
2	2	1	2	1	1
1	2	2	3	1	1
2	1	1	1	2	3
2	1	3	4	2	3
2	1	5	7	2	3

If only four registers are available then one may (with some care) economize by keeping C and D side by side in a register $\bar{C}$ and similarly E and F in a register $\bar{E}$.
Operation I now reads:
We put A/B in A and $\bar{C} + \bar{E}$ in $\bar{C}$.
Operation II now reads:
We interchange A and B, $\bar{C}$ and $\bar{E}$.
The example now reads:

	A	B	C	E
	10	2	.00010000	.00000001
	5	2	.00010001	.00000001
	2.5	2	.00010002	.00000001
	.....	.....	.....	.....
	1.0010415	1.0043363	.06432136	.01460485
Op. II→				
	1.0043363	1.0010415	.01460485	.06432136

We now split .06432136 into two parts and divide: $\log_{10} 2 \approx 0643/2136 = .30102996$. Of course the "$D$" and "$F$" parts of the numbers must not be allowed to overlap the first halves.

The simplicity of the rules given in either pair of *Operations I, II* is a recommendation for use of this logarithm in automatic computing machines. So also is the fact mentioned above that for "almost all" a_0 and a_1, the rate of convergence is essentially independent of these numbers and is, on the average, about one decimal place per complete cycle (that is, for each *Operation II*). We justify this claim as follows. KHINTCHINE[2] has shown that for "almost all" real numbers, $0 < x < 1$, the regular continued fraction has the property that the geometric mean of the n's is an absolute constant. This number, known as Khintchine's constant, has been computed by the author to ten figures[3]. Its numerical value is approximately

$$K = 2.685452001.$$

Now we saw in (2) that the n's regulate the rate of growth of the Q's. To estimate this growth we assume that the Q's are governed (in the mean) by the linear difference equation:

$$Q_{i+1} = Q_{i-1} + KQ_i.$$

From the theory of such equations, we find that for large i:

$$Q_i \doteq a\{\tfrac{1}{2}(K + [4 + K^2]^{\frac{1}{2}})\}^i$$

Now it is easily seen that the error in any approximation P_i/Q_i is of the order of $(Q_i)^{-2}$ and therefore, per average cycle, the error should decrease by a factor of $4\{K + (4 + K^2)^{\frac{1}{2}}\}^{-2} \doteq 1/9.1$ or approximately one decimal place.

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[1] See for example, D. TEICHROEW, "Use of continued fractions in high speed computing," *MTAC*, v. 6, 1952, p. 127.
[2] A. KHINTCHINE, "Zur metrische Kettenbruchtheorie," *Compositio Math.*, v. 3, 1936. p. 276–285.
[3] DANIEL SHANKS, "Note on an absolute constant of Khintchine," *MTAC*, v. 4, 1950, p. 28.

The Product Form for the Inverse in the Simplex Method

Summary: When a matrix is represented as a product of "elementary" matrices, the matrix, its transpose, its inverse and inverse transpose are readily available for vector multiplication. By an "elementary matrix" is meant one formed from the identity matrix by replacing one column; thus an elementary matrix can be compactly recorded by the subscript of the altered column and the values of the elements in it. In the revised simplex method,[1] both the inverse and inverse transpose of a "basic" matrix are needed; more significant, however, is the fact that each iteration replaces one of the columns of the basis. In the product form of representation, this change can be conveniently effected by multiplying the previous matrix by an elementary matrix; thus, only one additional column of information need be recorded with each iteration. This approach places relatively greater emphasis on "reading" operations than "writing" and thereby reduces computation time. Using the I.B.M. Card Programmed Calculator, a novel feature results: when the inverse matrix is needed at one stage and its transpose at another, this is achieved simply by turning over the deck of cards representing the inverse.

Introduction:

The simplex method is an algorithm for determining values for a set of n non-negative variables which minimizes a linear form subject to m linear restraints.[1,2a,3] It may be characterized briefly as a finite iterative procedure. Each iteration produces a new special solution to the restraint equations involving a subset of m of the variables, only one element of the subset changing on successive iterations; the remaining $n - m$ variables are equated to zero. The vectors of coefficients corresponding to the subset of m variables are linearly independent and constitute a *basis* in m-dimension real vector space. In the original simplex method[2a] (as coded for the SEAC[4] or as found

in CHARNES *et al.*[3], it is required that all the coefficient vectors be represented in terms of the latest basis; since the changes of basis are step-wise, a simple recursion relation suffices to alter the representations on each iteration.

The revised simplex method[1,5] differs from the original method in that it uses the same recursion relations to transform only the inverse of the basis for each iteration. It has been introduced to reduce the quantity of writing at each step (which it does in general), and is particularly effective for linear programming models where the original matrix of coefficients is largely composed of zeros, as for example, in the transportation model[2c] or dynamic economic and production models.[2b] If the original method is used, these zeros would be replaced by non-zeros in the successive iterations and this greatly increases the computational effort. On the other hand, the revised method leaves those zeros intact.

One important feature of the simplex method is concerned with the criteria by which one of the vectors in the basis is replaced by a vector not in the basis to form the basis of the next iteration. When the constant terms of the restraint equations are not general, *the choice of the vector to drop from the basis may be ambiguous* and an arbitrary selection (as pointed out in unpublished examples by ALAN HOFFMAN and PHILIP WOLFE) may lead to non-convergence. Several devices exist, however, for perturbing the constant terms so as to avoid this difficulty. The earliest proposal along these lines[2a] consisted in modifying the vector of constant terms by a specially weighted combination of the unit vectors. This approach may be used conveniently both for the revised and original simplex methods[1,5]. With the original simplex method, there is another natural way to form the *pertubation* which consists in adding a weighted linear combination of the column vectors to the vector of constant terms. This was suggested first by ORDEN and developed independently by CHARNES[3].

Although considerable attention has been paid to the above difficulty (called *degeneracy*), *it usually does not lead to non-convergence.* The type of problems in which it can cause non-convergence appear to be exceedingly rare. To date, there have been only two examples and these were artificially constructed for this purpose. Accordingly, the SEAC code and the RAND code use an *arbitrary* selection criteria in case of ambiguity. In these codes, a deliberate decision was made to use a simple code in lieu of a more complex one needed to cover a possible case that may *never* arise in practice.

The present method of using the product form for the representation of the inverse of the matrix, also makes use of this simplification. Again, provision could be made for covering the rare non-convergent case, but again, it does not appear to be worth-while.

We shall now describe a process by which a square non-singular matrix may be expressed as a product of elementary matrices of the form (2) below. This is illustratively seen for the simplex process which involves a step-wise change of basis matrix, that is to say, two successive matrices differ by only one column. Using a notation consistent with[1,5] let $B^{(l-1)} = (P_0, P_{j_1}, \cdots, P_{j_m})$ denote the $(l-1)^{\text{th}}$ basis. If, in the next basis, P_s is to replace P_{j_r}, then it is easy to show that

$$[B^{(l)}]^{-1} = E_l[B^{(l-1)}]^{-1}, \tag{1}$$

where E_l and E_l^{-1} are elementary matrices related by

$$(2)\quad E_l = [U_0,\cdots, U_{r-1}, \eta_l, U_{r+1},\cdots, U_m] = [U_0,\cdots, U_{r-1}, Y_l, U_{r+1},\cdots, U_m]^{-1},$$

where U_i is a unit vector with unity in the $(i+1)^{st}$ component, η_l is a vector whose components η_{il} are related to components y_{il} of Y_l by

$$(3)\quad \begin{aligned} \eta_{il} &= -\, y_{il}/y_{rl}, \qquad i \neq r \\ \eta_{rl} &= 1/y_{rl}, \end{aligned}$$

where it is necessary that $y_{rl} \neq 0$ and Y_l is defined by

$$(4)\quad Y_l = [B^{(l-1)}]^{-1}P_s.$$

Successive applications of (1) for $l = k, k-1, k-2,\cdots, 1$ yield

$$(5)\quad (B^{(k)})^{-1} = E_kE_{k-1}\cdots E_1[B^{(0)}]^{-1},$$

where $B^{(0)}$ is the initial basis. It is usually easy to arrange that the *initial basis* $B^{(0)}$ *be the identity matrix so that* $B^{(0)}$ may be dropped from (5).

Consider the problem of computing a row vector β_0, defined by

$$(6)\quad \beta_0 = \alpha B^{-1} = \alpha E_kE_{k-1}\cdots E_1,$$

where α is a given row vector (actually a unit vector[1,5]). Such a vector is required by the revised simplex method as the first step in determining the vector P_s to introduce into the basis. It is clear that β_0 can be obtained by successive transformations on row vectors, i.e., forming $(\alpha)E_k$, $(\alpha E_k)E_{k-1}$, $\cdots$, etc. However, when a row vector $A = (a_0, a_1,\cdots, a_m)$ is transformed into a row vector $B = (b_0, b_1,\cdots, b_m)$ by multiplying A on the right by an elementary matrix E_l one obtains simply

$$(7)\quad \begin{aligned} b_i &= a_i, \qquad i \neq r_l \\ b_{r_l} &= \sum_0^m \eta_{il}a_i, \end{aligned}$$

where, because r may be different for different l, we have set $r = r_l$.

Consider next the problem of computing Y by relation (4).

$$(8)\quad Y = B^{-1}P_s = E_kE_{k-1}\cdots E_1P_s.$$

It is clear that Y can be determined by successive transformations on column vectors, i.e., forming $E_1(P_s),\cdots$, etc. However, when a column vector $C = \{c_0, c_1,\cdots, c_m\}$ is transformed into a column vector $D = \{d_0, d_1,\cdots, d_m\}$ by multiplying C on the left by a matrix of the special form E, one obtains simply

$$(9)\quad \begin{aligned} d_i &= c_i + \eta_{il}c_{r_l}, \qquad i \neq r_l \\ d_{r_l} &= \qquad \eta_{r_l l}c_{r_l}. \end{aligned}$$

From (7) and (9) it is clear that the only essential information contained in E_l is the set of values η_{il} and the index r_l. Note further that in (8), the successive E_l are used with *increasing* l and it follows from (9) that it *is necessary* to know r_l *before* using the η_{il}. On the other hand, in (6), the E_l are used in *decreasing* sequence of l but from (7) it *is not necessary* to know r_l until *after* the η_{il} have been used. The perfect complementarity of the preceding two sentences, together with the fact that $\sum_0^m \eta_{il}a_i$ can obviously be computed starting with $i = m$ as well as with $i = 0$, makes it clear that (6) may be computed using the information in the reverse order of that used in (8).

Let L_l denote the ordered set of 'words' of information

$$L_l = \{r_l; \eta_{0l}, \eta_{1l}, \cdots, \eta_{ml}\}. \tag{10}$$

Then each change of a column of B will produce a new L_{l+1} which may be stored in consecutive order to the previously computed $L_1, L_2, \cdots, L_l$.

On the CPC, by punching two sets of instructions on each card—one being, in form, the reflection, in the vertical center line, of the other (with appropriate adjustments for difference in algorithms (7) and (9))—the transpose use of the inverse may be accomplished by simply turning the cards over using the vertical center line of the card for the axis.

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[1] G. B. Dantzig, Alex Orden & Philip Wolfe, *The Generalized Simplex Method.* Rand P-392-1 dated August 4, 1953.

[2] T. C. Koopmans, Ed., *Activity Analysis of Production and Allocation.* New York, 1951.
(a) George B. Dantzig, *Maximization of a Linear Function of Variables Subject to Linear Inequalities.* P. 339–347.
(b) *The Programming of Interdependent Activities: Mathematical Model.* P. 19–32.
(c) *Application of the Simplex Method to a Transportation Problem.* P. 359–373.
(d) T. C. Koopmans & S. Reiter, *A Model of Transportation.* P. 222–259.

[3] A. Charnes, W. W. Cooper & A. Henderson, *An Introduction to Linear Programming.* New York, 1953.

[4] A. Hoffman, M. Mannos, D. Sokolowsky & N. Wiegmann, "Computational experience in solving linear programs," Soc. Industrial and Applied Math., *Jn.*, v. 1, 1953, p. 17–33.

[5] G. B. Dantzig, *Computational Algorithm of the Simplex Method.* Rand P-394, April 10, 1953.

On Modified Divided Differences II

[Continued from *MTAC*, v. 8, p. 1–11]

Errors of Type (c). A question that presents itself is the extent to which errors of Type c will mask an isolated error. It will be desirable to approach the problem from a statistical standpoint, and to introduce the simplifying assumptions that the errors of Type (c) behave like round-off errors, subject to the following restrictions:

(a) The errors e_k in the entries u_k all have the same, uniform distribution between $-\frac{1}{2}a + c$ and $\frac{1}{2}a + c$, where a and c are fixed constants. Thus if entries are rounded to a fixed number of decimal places, the assumption is that the rounding will range uniformly between $\pm\frac{1}{2}$ units of the last place; that is, c is zero. Or, if the entries are "chopped"—that is, all digits beyond a fixed decimal place are dropped, without rounding, then the assumption is that the error in the last place ranges between 0 and unity; that is, $c = \frac{1}{2}$. It will be shown that the distribution function is independent of c, provided c is the same for all u_k.

(b) The errors in the tabular entries are independent of one another. The conditions (a) and (b) constitute a useful idealized model.

For the case of equally-spaced arguments the distribution function of the round-off error in differences of the first, second, and third order has been given explicitly by LOWAN & LADERMAN[1], and the method can be used for obtaining the distribution function for differences of all orders. A somewhat more elaborate study has been published by A. VAN WIJNGAARDEN[2]. We shall here follow the method of Lowan and Laderman, based on Fourier transforms. Consider the sum

$$w_n = A_0e_0 + A_1e_1 + \cdots + A_ne_n, \tag{2.16}$$

where the coefficients A_k are constants, and all the values of e_k are subject to the restrictions (a) and (b). Let $f(w, x)dx$ denote the probability element of the distribution; that is, $\int_{-\infty}^{t} f(w,x)dx = F(w,t)$ is the distribution function of w. For the case when $c = 0$ in condition (a), we have

$$f(e_k, x) = \frac{1}{a}, \text{ if } -\frac{1}{2}a \leq x \leq \frac{1}{2}a;\, f(e_k, x) = 0, \text{ if } |x| > \frac{1}{2}a. \tag{2.17}$$

$$f(Ae_k, x) = \frac{1}{a|A|}, \text{ if } -\frac{1}{2}a|A| < x < \frac{1}{2}a|A|, \tag{2.18}$$

$$f(Ae_k, x) = 0,\ |x| > \frac{1}{2}aA, \text{ for constant } A.$$

The characteristic function $g(w, t)$ associated with a distribution function $F(w, t)$ is defined by

$$g(w, t) = \int_{-\infty}^{\infty} e^{itx} f(w, x)dx, \tag{2.19}$$

and by the Fourier inversion theorem

$$f(w, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\, g(w, t)dt. \tag{2.20}$$

It is known that the characteristic function associated with the distribution of the sum of n independent variables is the product of the characteristic

functions associated with the distributions of the n individual variables. This gives, for $w = Ae_k$,

$$(2.21) \qquad g(w, t) = \frac{1}{a|A|}\int_{-\frac{1}{2}a|A|}^{\frac{1}{2}a|A|} e^{itx}\,dx = \frac{\sin\left(\frac{1}{2}aAt\right)}{\frac{1}{2}aAt}.$$

Hence for w_n defined in (2.16)

$$(2.22) \qquad g(w_n, t) = \prod_{k=0}^{n} \frac{\sin\left(\frac{1}{2}aA_kt\right)}{\frac{1}{2}aA_kt}.$$

Using (2.20), the frequency function for w_n is given by

$$(2.23) \qquad \begin{aligned} f(w_n, x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} \prod_{k=0}^{n} \frac{\sin\left(\frac{1}{2}aA_kt\right)}{\frac{1}{2}aA_kt}\,dt \\ &= \frac{1}{\pi}\int_{0}^{\infty} \frac{\cos(tx)}{t^{n+1}} \prod_{k=0}^{n} \frac{\sin\left(\frac{1}{2}aA_kt\right)}{\frac{1}{2}aA_k}\,dt. \end{aligned}$$

The probability that w_n takes on a value between b and c, $c > b$, is then given by

$$\int_b^c f(w_n, x)dx.$$

It is to be noted that the integrand in (2.23) is unchanged when A_k is replaced by $-A_k$; it will therefore be convenient to write

$$(2.24) \qquad \begin{cases} \beta_k = a|A_k| \\ G(x, t) = \cos tx \cdot \displaystyle\prod_{k=0}^{n} \frac{\sin\left(\frac{1}{2}\beta_k t\right)}{\frac{1}{2}\beta_k}. \end{cases}$$

Let

$$G^{(k)}(x, t) = d^kG(x, t)/dt^k; \quad G^{(0)}(x, t) = G(x, t).$$

Then

$$G^{(k)}(x, t) = t^{n+1-k}V(x, t), \quad 0 \le k \le n + 1,$$

where $V(x, t)$ is bounded for all t. Moreover $G^{(k)}(x, \infty)$ is bounded. Integrate (2.23) by parts. This gives

$$(2.25) \qquad f(w_n, x) = \frac{1}{\pi}\left[\frac{-G(x, t)}{nt^n}\right]_0^{\infty} + \frac{1}{\pi}\int_0^{\infty} \frac{G^{(1)}(x, t)}{nt^n}\,dt.$$

The term within the bracket vanishes at both limits and only the integral remains on the right-hand side of (2.25). Repeat the process of partial integration until we arrive at

$$(2.26) \qquad f(w_n, x) = \frac{1}{\pi}\int_0^{\infty} \frac{1}{n!}\frac{G^{(n)}(x, t)}{t}\,dt.$$

By expanding $G(x, t)$ into a sum of sines (or cosines) and performing the differentiations with respect to t, it can be verified that $f(w_n, x) = 0$ if $|x| \ge \frac{a}{2}\sum|A_k|$, as it should. In the case where the numbers A_k are the binomial

coefficients $(-1)^k \binom{n}{k}$, w_n represents the error in the nth ordinary difference of u_n, due to the individual errors e_k. In the case of modified divided differences, the numbers A_k represent $M_{n,k+n}$ in even central differences or corresponding coefficients in differences of odd order. It should be observed that in all divided differences, whether modified or not,

$$\sum_r M_{k,r} = 0. \tag{2.27}$$

For the coefficients $M_{k,r}$ are independent of the function u_k. In the special case when $u_k = 1$, the differences must vanish; hence (2.27). As a consequence of (2.27) it is clear that if $e_k = c + \delta_k$, where δ_k is uniformly distributed with mean zero, the constant c drops out in the sum represented by w_n. Hence even when entries have been "chopped" it is sufficient to study the distribution functions associated with δ_k.

For purpose of comparison, the known results for ordinary differences of orders up to three are summarized below, along with some of the corresponding distributions for modified divided differences.

First modified divided difference:

$$w_1 = \frac{w}{x_1 - x_0}(e_1 - e_0) = \frac{e_1 - e_0}{c_1};$$

$$f(w_1, t) = \begin{cases} (c_1/a)^2 \left[\dfrac{a}{c_1} - |t|\right], & -\dfrac{a}{c_1} \le t \le \dfrac{a}{c_1} \\ 0 \text{ elsewhere.} \end{cases} \tag{2.28}$$

If $c_1 = 1$, we have the frequency function for the first ordinary difference.

Second modified divided difference:

$$w_2 = M_{0,-1}e_{-1} + M_{0,0}e_0 + M_{0,1}e_1 = -\frac{2e_0}{c_1c_0} + \frac{2e_1}{c_1(c_0 + c_1)} + \frac{2e_{-1}}{c_0(c_0 + c_1)}.$$

Assume $c_0 \le c_1$

$$f(w_2, t) = \begin{cases} \dfrac{2a^2c_0c_1 - \left[c_0c_1\left(\dfrac{c_0 + c_1}{2}\right)\right]^2 t^2}{4a^3}, & \text{if } 0 \le |t| \le \dfrac{2a}{c_1(c_0 + c_1)} \\ \dfrac{c_0}{4a^2}\left[-|t|c_0c_1(c_0 + c_1) + ac_0 + 2ac_1\right], & \text{if } \dfrac{2a}{c_1(c_0 + c_1)} \le |t| \le \dfrac{2a}{c_0(c_1 + c_0)} \\ \left(\dfrac{c_0 + c_1}{2}\right)^2 \left[\dfrac{(tc_0c_1)^2 - 4|t|c_0c_1a + 4a^2}{4a^3}\right], & \text{if } \dfrac{2a}{c_0(c_0 + c_1)} \le |t| \le \dfrac{2a}{c_0c_1} \\ 0 \text{ elsewhere.} \end{cases} \tag{2.29}$$

Thus $f(w_2, x)$ consists of five curves, symmetric about the y-axis, two of those curves being linear. If $c_0 = c_1 = 1$, each line shrinks to a point, and $f(w_2, t)$ consists of three parabolas.

Third ordinary difference: The following table is taken from LOWAN & LADERMAN[1]:

$$
(2.30)\qquad f(w_3, t) = \begin{cases} \dfrac{1}{a^4}\left[\dfrac{|t|^3}{27} - \dfrac{at^2}{9} + \dfrac{8a^3}{27}\right], & 0 \le |t| \le a \\[2ex] \dfrac{1}{a^2}\left[-\dfrac{|t|}{9} + \dfrac{a}{3}\right], & a \le |t| \le 2a \\[2ex] \dfrac{1}{a^4}\left[\dfrac{|t|^3}{54} - \dfrac{at^2}{9} + \dfrac{a^2|t|}{9} + \dfrac{5a^3}{27}\right], & 2a \le |t| \le 3a \\[2ex] \dfrac{1}{a^4}\left[-\dfrac{|t|^3}{54} + \dfrac{2at^2}{9} - \dfrac{8a^2|t|}{9} + \dfrac{32a^3}{27}\right], & 3a \le t \le 4a \\[2ex] 0 \text{ elsewhere.} \end{cases}
$$

There are seven curves in $f(w_3, t)$, two of them being straight lines. Let us consider the frequency function for the third modified divided difference, and write for brevity

$$
w_3 = \sum_{k=0}^{3} A_k e_k; \quad B_k = a|A_k|.
$$

Making use of (2.27), we can put $A_3 = -(A_0 + A_1 + A_2)$. If the arguments x_k form an increasing sequence, the *sign* of A_k is independent of the magnitude of the intervals $(x_k - x_{k-1})$; hence the sign will be the same as in the ordinary third difference, and it is permissible to write

$$
|A_3| = |A_2| - |A_1| + |A_0|.
$$

$$
G(x, t) = \frac{\cos xt[\sin(\tfrac{1}{2}B_0 t)\sin(\tfrac{1}{2}B_1 t)\sin(\tfrac{1}{2}B_2 t)\sin(\tfrac{1}{2}(B_2 - B_1 + B_0)t]}{[B_0B_1B_2(B_2 - B_1 + B_0)]/16}.
$$

By expanding the above into a sum of sines and cosines, it can be verified that

$$
(2.31)\qquad G^{(3)}(x, t) = C\sum_{k=1}^{8}\{b_k(x + D_k)^3 \sin[(x + D_k)t] + b_k(x - D_k)^3 \sin[(x - D_k)t]\}.
$$

In (2.31) C is some constant, $b_k = \pm 1$ if $D_k \neq 0$, $b_k = \tfrac{1}{2}$, if $D_k = 0$, and D_k assumes the following values:

$$
(2.32)\qquad \begin{aligned} &D_1 = 0, D_2 = B_2 - B_1, D_3 = B_0, D_4 = (B_2 - B_1 + B_0), \\ &D_5 = (B_1 - B_0), D_6 = B_1, D_7 = B_2, D_8 = B_2 + B_0. \end{aligned}
$$

Let us consider the special case when

$$
D_1 \le D_2 \le D_3 \le D_4 \le D_5 \le D_6 \le D_7 \le D_8 \le B_0 + B_1 + B_2,
$$

and let x be positive. Since

$$\frac{1}{\pi}\int_0^\infty (\sin bt/t)dt = \tfrac{1}{2} \text{ if } b > 0, \quad -\tfrac{1}{2} \text{ if } b < 0, \text{ and } 0 \text{ if } b = 0,$$

the terms of (2.31) involving $(x + D_k)^3$ will contribute, after integration, terms that have the same sign for all positive values of x between 0 and $(B_0 + B_1 + B_2)$. On the other hand, the terms involving $(x - D_k)^3$ will vary in sign, depending on whether x is greater than or less than D_k. Clearly x can fall into any one of the eight regions separated by the inequalities of (2.32), and in the most general case when no two D_k are equal, the set of terms of (2.31) will comprise, after integration, cubic polynomials (or polynomials of lower degree) with coefficients that will be different in the several regions. Since $f(w_3, x)$ is an even function of x, the curve comprising the eight arcs will be reflected through the y-axis, with a common central arc. There will therefore be 15 arcs to the frequency function for the most general values of D_k. These shrink to seven arcs in the case of ordinary differences. The labor of computing the exact distribution seems to be prohibitive, and alternative approximations will be considered.

It is known that for large enough n, $F(w_n, t)$ tends to approach the normal distribution with mean 0 and standard deviation $\sigma(w_n)$, where

$$\sigma(w_n) = \left(\sum_{k=0}^{n} A_k^2\right)^{\frac{1}{2}} \sigma(e_k) = a\left(\sum_{k=0}^{n} A_k^2/12\right)^{\frac{1}{2}}.$$

In the above, $\sigma(e_k)$ is the standard deviation of e_k. It will be instructive to examine the probabilities that w_3, associated with the third ordinary difference, will fall into certain intervals, and to compare them with the probabilities obtained from the corresponding normal distribution. For w_3, $\sigma(w_3) = 1.29099a$. The following schedule lists some calculations:

Range of w_3	Theoretical Probability Based on (2.30)	Probability Based on Normal Distribution
$2\ \sigma \leq \lvert w_3 \rvert \leq 4a$	.03687	.04350
$2.4\sigma < \lvert w_3 \rvert \leq 4a$	.00612	.01440
$2.5\sigma \leq \lvert w_3 \rvert \leq 4a$	.00331	.01236

It is to be observed that in the third ordinary difference, the normal distribution exaggerates the area of the "tail"-end of the distribution. However, the discrepancy between the two distributions is no worse than by a factor of 2.4 for the first two ranges of the schedule, and agreement is expected to be closer in higher differences.

Assuming that the probability of w_1 being numerically greater than 2.4σ is approximately the same in higher differences, it might be reasonable to tolerate a discrepancy of 2.4σ. Such a discrepancy is expected to occur once in 163 listings of the third difference, according to the exact distribution, and if we use the normal distribution as a guide, it may occur once in 69 entries in differences of high order. It is costly to examine too many doubtful entries; often it is much more advantageous to calculate two added decimal

places in the entries, so that discrepancies with even higher probability can be passed. Much depends on the problem at hand.

The following schedule lists the value of 2.4σ in differences of orders n up to 10. Corresponding values of $.4\sum|A_k|a$ are also included; for some values of n, $.3\sum|A_k|a$ is also tabulated. A_k is the binomial coefficient $\binom{n}{k}$.

n	2.4σ	$.4\sum\lvert A_k\rvert a$	n	2.4σ	$.4\sum\lvert A_k\rvert a$	$.3\sum\lvert A_k\rvert a$
2	$1.697a$	$1.6a$	7	$40.588a$	$51.2a$	$38.4a$
3	$3.098a$	$3.2a$	8	$78.598a$	$102.4a$	$76.8a$
4	$5.797a$	$6.4a$	9	$152.766a$	$204.8a$	$153.6a$
5	$10.998a$	$12.8a$	10	$297.797a$	$409.6a$	$307.2a$
6	$21.060a$	$25.6a$				

It should be observed that the simpler function $.4\sum|A_k|a$ is close in magnitude to 2.4σ if $n \leq 6$ and can be used in its place as a basis for a reasonable tolerance. In differences of higher order $.3\sum|A_k|a$ is close to 2.4σ.

To what extent can the above schedule be used as a basis for establishing a reasonable tolerance for modified divided differences? One way is to compute a few values of $M_{k,k+r}$ as a basis for estimating σ. That may be laborious. Some qualitative estimates can be obtained from the form of (1.12). Let A_k denote the coefficient of u_k in the ordinary difference of order $2n$. Then it is clear that

$$M_{0,k}/A_k = \prod_{\substack{j=-n\\k\neq j}}^{n} \left(\frac{k-j}{p_k - p_j}\right) = \prod_{\substack{j=-n\\k\neq j}}^{n} \left(\frac{|k-j|}{c_s + c_{s-1} + \ldots + c_{t-1}}\right),$$

where s is the greater of k and j and t is the smaller. Thus in a region where the values of c_k are consistently greater than unity, the standard deviation is *lower* than that in the ordinary difference, and the tolerance should be more stringent than for ordinary differences. On the other hand, if the c_k are all *smaller* than unity in a region, then σ is larger. Whe-ever the standard deviation for ordinary differences can be used as a basis, the schedule of 2.4σ offers one means of judging the extent to which an isolated error will be masked by smaller errors in neighboring entries.

Secondary Effects in Round-offs. When ordinary differences are considered, all the round-off errors occur in the entires u_k, and the process of taking differences introduces no other errors. In forming divided differences, however, (modified or not) multiplications and divisions are involved, and the result is rounded off to a fixed number of decimals. Hence the cumulative effect of such roundings must be considered, and we must specify the order in which various operations are to be performed. Let $g(k, r)$ denote for brevity the rth modified divided difference associated with t_k. Then $g(k, r)$ is formed as follows:

$$g(k, r) = y(k, r)[g(k+1, r-1) - g(k, r-1)],$$

where

$$y(k, r) = rw/[t_{k+\frac{1}{2}r} - t_{k-\frac{1}{2}r}], \quad \text{if } r \text{ is even}$$

$$y(k, r) = rw/[t_{k+\frac{1}{2}(r+1)} - t_{k-\frac{1}{2}(r-1)}], \quad \text{if } r \text{ is odd.}$$

The differences are useful for interpolation or error detection only in the case where successive differences fall off in magnitude. Let $y(k, r)$ be computed to the maximum attainable accuracy (depending on the computing machine), and then multiplied by $g(k + 1, r - 1) - g(k, r - 1)$, which in successive differences is expected to have fewer significant figures than $y(k, r)$ in the useful case. Thus the error due to carrying an inexact $y(k, r)$ is expected to be of a lower order of magnitude than the error in $g(k, r)$ and we shall neglect its consideration. The principal new error is therefore the rounding of $g(k, r)$ to a fixed number of decimals. Thus in each successive difference there is introduced a new rounding ρ_j, assumed to satisfy conditions (a) and (b). If w_n is the error function in the nth difference with exact operations in obtaining $g(k, r)$ then the total error due to all types of roundings is

$$V_n = w_n + w_{n-1} + w_{n-2} + \ldots + w_1 + \rho_n,$$

where ρ_j replaces e_j in w_{n-j}, $j > 0$.

It is necessary to have an estimate of V_n, as compared with w_n. Let us consider the case where the standard deviation of w_n is used as an estimate, and let

$$\phi_n = a^2 \sum A_k^2/12; \quad \sigma_n = \phi_n^{\frac{1}{2}}.$$

Then the standard deviation of V_n is

$$\sigma_{V_n} = \left(\frac{1}{12} a^2 + \sum_{k=1}^{n} \phi_k \right)^{\frac{1}{2}}.$$

In the special case where the A_k are binomial coefficients (i.e., for the ordinary difference) it can be verified that

$$\sigma_{V_n} = d_n \sigma_n$$

where $d_2 = 1.155$, $d_3 = d_4 = 1.183$, and for n ranging between 4 and 10, d_n decreases steadily up to 1.165 for $n = 10$. Hence if a tolerance of $2.4\sigma_n$ is allowed for ordinary differences, a tolerance of $2.8\,\sigma_n$ should be reasonable for divided differences. If storage space permits, it is of course possible to carry two more decimals in the divided differences than in the function, so as to lessen the rounding error. If that is done, the added tolerance is not necessary.

The writer acknowledges gratefully the help given by Dr. J. LADERMAN, who read the manuscript and offered many valuable suggestions.

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[1] ARNOLD N. LOWAN & JACK LADERMAN, "On the distribution of errors in nth tabular differences," *Annals of Math. Statistics*, v. 10, 1939, p. 360–364. One minor detail of the paper is faulty, but the results are correct. In evaluating integrals involving products of sines or cosines, it is stated, for example, that

$$\int_{-\infty}^{\infty} dt[\sin at]/t^{2n+1} = [(-1)^n a^{2n}/(2n)!] \int_0^{\infty} dt \sin at/t.$$

This of course is not true, since the left-hand integral does not exist if n is different from zero. However, it turns out that the contribution from the region around the origin in a sum of such integrals vanishes, and the result is right.

[2] A. van Wijngaarden, Afrondingsfouten MR3, Tevens ZW-(1950)-.001. Math. Centrum Rekenfdeling, Amsterdam (in Dutch). See also A. M. Ostrowski. *Two Explicit Formulae for the Distribution Function of the sums of n Uniformly Distributed Independent Variables*. Archiv d. Math., v. 3, 1952, p. 3–11.

RECENT MATHEMATICAL TABLES

1165[B,F].—H. S. Uhler, "On the 16th and 17th Perfect Numbers," *Scripta Math.*, v. 19, 1953, p. 128–131.

This note contains exact values of $2^{n-1}(2^n - 1)$ for $n = 2203$ and 2281, numbers of 1327 and 1373 decimal digits. These are the 16th and 17th perfect numbers. Exact values are given also of 2^n for $n = 560, 2202, 2280$ and those digits of 2^{4405} and 2^{4561} which are not identical with the corresponding digits of the perfect numbers mentioned above.

The author has informed the reviewer of the fact that the 1023rd digit was printed incorrectly: *for* 32633 *read* 32638. This substitution occurred between page proof and printing and would have gone undetected by any author but one having Uhler's indefatigable perspicacity.

D. H. L.

1166[C].—NBSCL, *Tables of* 10^x (*Antilogarithms to the Base* 10). NBS Applied Math. Series, No. 27, U. S. Gov. Printing Office, Washington, 1953, viii + 543 p., 19.3 × 26.0 cm. Price $3.50.

The main table in the work is Table I, a 500 page table of 10^x for $x = 0(.00001)1$. These 100000 values are given to 10D. The arrangement is in four columns of 50 pairs $(x, 10^x)$ each so that consecutive entries lie one under the other making linear interpolation easy. All eleven digits of 10^x are given in each entry. No differences are given. Linear interpolation gives 9D accuracy. The effect of the second difference on the 10th decimal may be read from a chart on p. vi. This amounts to at most 7 units in the 10th place.

Table II is a 15D radix table of 10^x. Specifically it gives 10^y where

$$y = n \cdot 10^{-p}, \quad n = 1(1)999, \quad p = 3(3)15.$$

From this table 14 figure antilogarithms can be found by multiplying five entries together. The table can also be made to serve as a table of common logarithms to 15D.

Table II is similar to that of Deprez[1] which gives 13D antilogarithms of $x = m \cdot 10^{-r}$ for

$$m = 1(1)999, \quad r = 7(3)13$$

in connection with a basic table for

$$x = 0(.0001)1.$$

Table II will be found very useful in connection with any ordinary radix type logarithm table for very precise work.

Table I is based on Dodson's[2] rare table of 1742. The entire Dodson table was transferred to punched cards and differenced on a tabulator. After correcting errors the table was checked by summing sets of 50 consecutive entries (as a geometric progression). Finally the printed page proof

was subjected to additional differencing. No list of errata in Dodson is given.

Table I is a handy companion to a table of 10 place logarithms since inverse interpolation is avoided in the usual passage from numbers to logarithms and back to numbers. However since interpolation is only linear (at least for 9D work), inverse interpolation presents no greater problem than direct interpolation, especially if one has even the smallest type of desk calculator. From this there are two alternative inferences: (a) One needs no table of antilogarithms; (b) one needs only a table of antilogarithms. Perhaps this table will appeal most to those who want to do no interpolation whatever.

This useful volume is the result of work done by the old New York Mathematical Tables Project.

D. H. L.

[1] F. Deprez, *Tables for Calculating, by Machine, Logarithms to 13 Places of Decimals.* Berne, 1939.
[2] J. Dodson, *Antilogarithmic Canon.* London, 1742.

1167[C].—NBSCL. *Table of Natural Logarithms for Arguments Between Zero and Five to Sixteen Decimal Places.* NBS Applied Math. Series, No. 31, U. S. Gov. Printing Office, Washington, 1953, [Reissue of MT 10], x + 501 p., 20.1 × 25.9 cm. Price $3.25.

The original NYMTP Table 10 [v. 3 of the original 4v.] $x = 0(.0001)\,5$ is hereby reissued to meet a continued demand. The preface promises also a reissue of the fourth volume for $x = 5(.0001)10$. Although the Introduction states that there has been no revision of the tabular content, there has been a change in the rule for the indication of the signs of the logarithms. Thus in the original edition the logarithm of .0184 is given as 3.995 . . . , the fact that this number is actually negative being understood. Now the minus sign is printed explicitly. This improvement is carried out with a single exception and this occurs at the very first real entry of the table where the logarithm of .0001 is given as 9.21034. . . .

One further change may be noted and this refers to the last entry in the table. The reader who is familiar with the NYMTP tables will recall that arguments are given at the bottom of the page without the corresponding functional values. This tantalizing procedure is followed in the present volume except at the very end where the editor has relented and has given

$$\ln 5.0000 = 1.6094379124341004.$$

Apparently there are no errata known in this monumental table of 1941.

D. H. L.

1168[F].—E. S. Barnes & H. P. F. Swinnerton-Dyer, "The inhomogeneous minima of binary quadratic forms," *Acta Math.*, v. 87, 1952, p. 259–320.

Let $f(x, y) = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form with real coefficients and discriminant $D = b^2 - 4ac > 0$. For any point P: (x_0, y_0), where x_0, y_0 are real, define $M(f;P)$ to be the lower bound of

$$|f(x + x_0, \quad y + y_0)|$$

taken for all points P and call $M(f)$ the upper bound of $M(f;P)$. Let C be the set of points P for which $M(f;P) = M(f)$ and $M_2(f)$ the upper bound of $M(f;P)$ over all P not belonging to C. Clearly $M_2(f) \leq M(f)$. If the strict inequality holds, $M(f)$ is called an isolated minimum.

The authors list in their table [p. 315–317] the values of $M(f)$ for forms $x^2 - my^2$ for all square-free $m \equiv 2$ or $3 \pmod 4$, $m \leq 101$, except $m = 46$, 67, 71, 86, 94 and many corresponding values of $M_2(f)$. In many cases complete sets of incongruent (mod 1) points are given for which $M(f;P) = M(f)$ or $M(f;P) = M_2(f)$. All minima given in the table are isolated minima.

A corresponding table is given for forms $f = x^2 + xy - \frac{1}{4}(m - 1)y^2$ where $m \equiv 1 \pmod 4$, m is square-free and not greater than 101, except $m = 57$ and 73.

Sources for the results listed, many in the accompanying paper, are given.

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1169[F].—A. Gloden, *Table des Solutions de la congruence* $x^4 + 1 \equiv 0 \pmod p$ *pour* $600000 < p < 800000$. Luxembourg, 1952, published by the author, rue Jean Jaurès, 11, Luxembourg, 22 p., 29.8 × 21.0 cm., mimeographed. Price 120 francs belges.

This table is identical with that described as UMT **158**, *MTAC* v. 7, p. 108 except that the appended factorizations there referred to are wanting.

1170[F,L].—A. van Wijngaarden. "On the coefficients of the modular invariant $J(\tau)$," K. Ned. Akad. v. Wetensch. *Proc.*, s.A, v. 56, 1953, p. 389–400.

Klein's fundamental modular invariant $J(\tau)$ has an expansion in the form

$$12^3 J(\tau) = x^{-1} + 744 + 196884x + 21493760x^2 + \cdots$$

$$= \sum_{n=-1}^{\infty} c(n)x^n$$

in which $x = q^2 = e^{2\pi i\tau}$. These coefficients are positive integers whose properties have only recently been investigated. The present paper gives a two page table of $c(n)$ for $n = -1(1)100$. These values are quite large, $c(100)$ having 53 digits, and the table represents a relatively large amount of computing. Previous tables are those of Berwick[1] for $n \leq 7$ and Zuckerman[2] for $n \leq 24$.

The intimate connections between $J(\tau)$ and other elliptic functions provide a variety of methods for the computation of $c(n)$. That none of these methods can be too easy is pointed out by the author who remarks that "the coefficients grow very rapidly with n and the digits have to come from somewhere."

Zuckerman exploited a connection between $c(n)$ and the number of partitions of $25n$. This becomes ineffective for $n = 25$ since Gupta's[3] tables of the partition function extend only to $n = 600$.

LEHMER[4] had proposed the formula

$$\sum_{k=-1}^{n} c(k)\tau(n-k) = 720\{91\sigma_{11}(n) + 600\tau(n)\}/691$$

and VAN DER POL[5] the more elegant formula

$$\sum_{k=-1}^{n} kc(k)\tau(n-k) = 24\sigma_{13}(n). \tag{1}$$

These formulas are based upon the connection between $J(\tau)$ and the Weierstrassian discriminant

$$\sum_{n=1}^{\infty} \tau(n)x^n = x \prod_{n=1}^{\infty} (1-x^n)^{24}$$

whose coefficients, known as Ramanujan's function, are now tabulated[6] to $n = 2500$. Formula (1) was used by the author up to $n = 50$. At this point another more elaborate method based on the relation

$$27J(\tau) = 2(\theta_2{}^8 + \theta_3{}^8 + \theta_4{}^8)(\theta_2{}^{-8} + \theta_3{}^{-8} + \theta_4{}^{-8})$$

between $J(\tau)$ and Jacobi's theta functions, was used to recompute and extend the table to $n > 100$. Various congruence properties of $c(n)$, including an interesting new one modulo 71, were used to check the calculations.

The table should be a valuable tool for further research on Klein's invariant.

D. H. L.

[1] W. E. H. BERWICK, "An invariant modular equation of the fifth order," *Quart. Jn. of Math.*, v. 47, 1916, p. 94–103.
[2] H. S. ZUCKERMAN, "The computation of the smaller coefficients of $J(\tau)$," Amer. Math. Soc., *Bull.*, v. 45, 1939, p. 917–919.
[3] H. GUPTA, "A table of partitions, I, II," London Math. Soc., *Proc.*, v. 39, 1935, p. 142–149, v. 42, 1937, p. 546–549.
[4] D. H. LEHMER, "Properties of the coefficients of the modular invariant $J(\tau)$," *Amer. Jn. of Math.*, v. 64, 1942, p. 488–502.
[5] B. VAN DER POL, "On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications, I, II," K. Ned. Akad. v. Wetensch., *Proc.*, s.A, v. 54, 1951, p. 261–284.
[6] See *MTAC*, v. 4, 1950, p. 162, UMT **101**.

1171[I].—H. E. SALZER, *Tables of Coefficients for the Numerical Calculation of Laplace Transforms*. NBS Applied Math. Series, No. 30, U. S. Gov. Printing Office, Washington, 1953, ii + 36 p., 20.1 × 25.9 cm. Price 25 cents.

These tables are intended to be used in the approximation of the transform

$$F(p) = \int_0^\infty e^{-pt}f(t)dt$$

by means of the sum

$$F(p) = \sum_{i=1}^{n-1} A_i f(i)$$

where

$$A_i = A_i^{(n)}(p)$$

depend on both p and n, the number of ordinates. The main tables are arranged by n which takes the values 2(1)11. The values of p are .1(.1)$n - 1$ up through $n = 7$. For $n = 8$ and 9 the interval is .2 and for $n = 10$ and 11 it is 1. Values of the A's are given to 9S. There are auxiliary tables as follows.

Table II (p. 27–36) gives values of $n!/p^{n+1}$ for $n = 0(1)10$, $p = .1(.1)10$ to 8s. This function is the Laplace transform of t^n and the table is intended for use in transforming polynomials up through those of degree 10.

The function $(n - 1)!p^nA_i^{(n)}(p)$ is a polynomial in p with integer coefficients. These polynomials are listed in the introduction (p. 7–8) together with the corresponding Lagrange interpolation coefficients (polynomials in t) for $n = 2(1)11$.

Four explicit examples are worked out including one in which the error is estimated.

These tables should be quite useful in dealing with functions whose Laplace transform is unfamiliar or intractable.

D. H. L.

1172[K].—Hirojiro Aoyama, "On a test in paired comparisons," Tokyo Inst. Math., *Annals*, v. 4, 1953, p. 83–87.

Let each of n persons, no two of which have the same occupation, rate his own occupation in comparison with each of the others. This gives rise to $k = \binom{n}{2}$ paired comparisons, to which we assign the following scores: +1 in those cases in which each subject in a pair rates his own occupation higher than the other, −1 in those cases in which each subject rates his own occupation lower than the other, 0 in all other cases. Let S be the total of the k scores. This is proposed as a test criterion for the null hypothesis that each subject is as likely as not to rate his own occupation higher than any other of the n occupations. Under this model for k throws of a pair of coins, S is the excess of the number of cases of both heads over the number of cases of both tails. A table is given for $\Pr(|S| \geqslant S_0)$ for $n = 3(1)9$ to at least 4D always giving at least 2S.

C. C. C.

1173[K].—Joseph Berkson, "A statistically precise and relatively simple method of estimating the bio-assay with quantal response, based on the logistic function," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 565–599.

This paper sets forth very clearly and succintly the author's "minimum logit chi-square" method in bio-assay. To facilitate the necessary computations three tables are provided. Table 1 gives $l = \ln [p/(1 - p)]$ to 5D for $p = .001\ (.001)\ .999$. Table 2 inverts Table 1, giving p to 5D for $l = 0(.01)$-4.99. Table 3 gives the weights needed in the logistic calculation, namely $p(1 - p)$ and $p(1 - p)l$, to 4D, for $p = 0(.001)1$.

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1174[K].—K. N. Chandler, "The distribution and frequency of record values," Roy. Stat. Soc., *Jn.*, s.B, v. 14, 1952, p. 220–228.

The frequency and interval of occurrence of record values is of considerable interest whether we are dealing with weather data or sampling other types of populations. The lowest record value at any given time is defined as that member of the sample which is less than or equal to all previous members and the greatest record value similarly is defined as that value which is greater than or equal to all previous values. In this paper Chandler considers the random series x_u, $u = 1, 2, 3$, etc. with the first record value X_1 equal to x_1. Let X_i be the first occurring value of x which is less than X_{i-1}, $i = 2, 3$, etc. Then X_r is called the rth lower record value (similar considerations apply to the higher record values). Assuming that the distribution function of x is either normal or rectangular, Chandler derives the probability distribution of the rth lower record value X_r (or rth higher record value), the distribution of the serial number, u_r, of the lower record value and also the probability distribution for the interval of occurrence (number of observations) between the rth lower record value and the $(r - 1)$st lower record value.

Table 1 and Table 2 of the paper give the .005, .01, .1, .5, .9, .99 and .995 probability points for the distribution of X_r for the normal distribution in standard units to 3D and for the rectangular distribution to 4S for $r = 2(1)9$ in both cases. Table 3 gives the probability that the serial number, u_r, of the lower record value, will be $\leqq n$ to 6D for $r = 3(1)9$ and $n = 3(1)30(5)60(10)100$, 200, 500, 1000, 2000, 5000, 10000, 20000, 50000, 100000, 200000, 500000, 1000000. Table IV gives the probability that the interval of occurrence, $u_r - u_{r-1}$, the number of observations between the rth lower record value and the $(r - 1)$st lower record value $\geqslant n$ to 6D for $r = 2(1)9$ and the same set of values of n as in Table 3. The serial numbers u_r, and their differences, $u_r - u_{r-1}$, do not have finite means and are independent of the parent population provided it has finite or zero probability density at all points!

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1175[K].—W. J. Dixon, "Power functions of the sign test and power efficiency for normal alternatives," *Annals Math. Stat.*, v. 24, 1953, p. 467–473.

The two sided sign test has long been used as a non-parametric test of significance. In using such a test one generally compares the number of changes in sign that occur in his observations with the expected number under a given hypothesis. In using such a non-parametric technique, one is interested in its power as well as how this power compares with the power of corresponding parametric tests. This paper contains tables which help the research worker answer such questions.

Tables I and II give respectively the power of the sign test to 5D for the 5% and 1% level of significance for sample sizes $N = 5(1)20(5)50(10)100$ in testing the null hypothesis that the proportion of objects in the population is .5 against varying proportions, $p = .05(.05).95$ in the alternative population.

To compare the power of the sign test with normal alternatives the author introduces a power efficiency function which gives the power efficiency of the sign test as compared with the normally based test for each alternative. Tables for selected values of the parameters and representative curves of this power efficiency function are given. The results exhibited by this function are compared with various asymptotic and approximate efficiency estimates that have been obtained by various authors in the past.

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1176[K].—BENJAMIN EPSTEIN & MILTON SOBEL, "Life testing," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 486–502.

For a characteristic X with density function

$$f(x;\theta) = (\exp(-x/\theta))/\theta$$

the maximum likelihood estimate of θ based on the first r order statistics of a sample of n is

$$\hat{\theta}_{r,n} = [x_{1,n} + \cdots + x_{r,n} + (n-r)x_{r,n}]/r. \quad (r \leqq n)$$

Furthermore, $2r\hat{\theta}_{r,n}/\theta$ is distributed as χ^2 with $2r$ degrees of freedom. Since this result is independent of n, an appreciable saving of time can be made in situations, such as life testing, where the observations become available in order by basing the estimate of θ on the first r available results from a larger sample. Table I tabulates to 2D the ratio $E(X_{r,n})/E(X_{r,r})$ of the expected time to obtain the first r results from a sample of n to the expected time to obtain all r results from a sample of r for $r = 1(1)5, 10$ and $n = 1(1)5(5)20$. A simple derivation of previously known forms for $E(X_{r,n})$ and $\mathrm{Var}(X_{r,n})$ is given.

It is also shown that the region of rejection for the best test of $H_1(\theta) = \theta_1$ against the alternative $H_2(\theta) = \theta_2 < \theta_1$ is given by $\hat{\theta}_{r,n} < C$. Table II tabulates for $\theta_1/\theta_2 = 1.5(.5)3(1)5(5)10$; $\alpha, \beta = .01$ and $.05$ the minimum value of r, and the corresponding upper and lower limits for C/θ_1 to 4D, such that the errors of Type I and Type II will be less than α and β, respectively. This table represents a rearrangement and, in some respects, an extension of tables published by EISENHART[1] which (in the notation of the present paper) gives for fixed r, α, and β the maximum value of θ_1/θ_2 such that the test conditions are satisfied. It is also directly related to other tables dealing with the operating characteristics of the one-sided tests of a hypothetical variance σ_0^2 against the alternatives $\sigma^2 < \sigma_0^2$.

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[1] C. EISENHART, M. W. HASTAY, W. A. WALLIS, editors, *Selected Techniques of Statistical Analysis*. New York and London, 1947, ch. 8, p. 267–318.

1177[K].—H. L. JONES, "Approximating the mode from weighted sample values," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 113–127.

This paper presents a method of estimating the population mode by a weighted sum of the order statistics of a sample. The derivation of the values

of the weights is based on an approximation to the maximum likelihood solution. The parametric form of the population is assumed to be known. The method is of a general nature but does not necessarily yield a reasonable approximation to the mode unless certain favorable conditions are satisfied. The case of a t-distribution with varying degrees of kurtosis is analyzed to illustrate application of the method. Table 1 contains weights to 2D to be used in approximating the mode of a t-distribution for n (sample size) = 3(1)10 and $\alpha_4 = \mu_4/\mu_2^2 = 3(.5)5, 6, 9, \infty$.

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1178[K].—R. KAMAT, "On the mean successive difference and its ratio to the root mean square," *Biometrika*, v. 40, 1953, p. 116–127.

Given a sequence of n normal variates $\{x_i\}$ with common mean and variance. The mean square successive difference is

$$d = (n - 1)^{-1} \sum_{i=1}^{n-1} |X_i - X_{i+1}|.$$

Table 1 presents the standard deviation, β_1 and β_2 of d/σ, for n = 3(1) 10(5) 30, 40, 50 to 4D. Table 2 presents approximate upper and lower .5, 1, 2.5 and 5% percentage points of d/σ, to 2D, using a Pearson Type I curve; exact results are given for $n = 3$.

The ratio $W = d/s$, where s is the sample standard deviation, is also considered. Table 4 presents the mean to 4D, the standard deviation to 4D, β_1 to 3D and β_2 to 2D of W for n = 5(5) 30, 40, 50. Upper and lower 0.5, 1, 2.5 and 5% points for W are given in Table 5 for n = 10(5)30(10)50 to 2D.

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1179[K].—TOSIO KITAGAWA, TEISUKE KITAHARA, YUKIO NOMACHI & NOBUO WATANABE, "On the determination of sample size from the two sample theoretical formulation," *Bulletin Math. Stat.*, v. 5, 1953, p. 35–46.

The authors consider a two-sample procedure alternative to that of Stein[1,2] for determining a confidence interval of fixed length $2d$ for the mean of a normal population with unknown variance. They give a table for the following function connected with the probability distribution of the second sample size:

$$I(n_2;n_1|d^2\sigma^{-2},\alpha,\beta) = \int_{b(n_2-1)}^{b(n_2)} \phi_{n_1}(s_1;\sigma)ds_1 \int_0^{c(n_2)} \phi_{n_2}(s_2;\sigma)ds_2,$$

where

$$b(n) = dn^{\frac{1}{2}}\{t_{n-1}(\alpha)\}^{-1}\{F_{n_1-1}^{n-1}\ (\beta)\}^{-\frac{1}{2}},$$
$$c(n) = dn^{\frac{1}{2}}\{t_{n-1}(\alpha)\}^{-1},$$

$t_{n-1}(\alpha)$ is the 100α-percentage point of the t-distribution with $n - 1$ degrees of freedom, $F_{n_1-1}^{n-1}\ (\beta)$ is the 100β-percentage point of the F-distribution with

$n - 1$ and $n_1 - 1$ degrees of freedom,

$$s_i = \left\{ (n_i - 1)^{-1} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \right\}^{\frac{1}{2}}, \quad i = 1, 2,$$

is the square root of the unbiased estimate of the population variance given by the first, respectively, second sample, of sizes n_1 and n_2, and $\phi_{n_i}(s_i;\sigma)$ denotes the density function of s_i. The tables are for only the selected combinations of the arguments $n_1 = 10, 15, 21, 25, 31$; $n_2 = 3(1)60$; $\alpha = .01, .05$; $\beta = .01, .05$; $d^2\sigma^{-2} = .75, .5, .25$.

JULIUS LIEBLEIN

NBSSEL

[1] W. G. COCHRAN, *Sampling Techniques*. New York, 1953, p. 59–60.
[2] B. M. SEELBINDER, "On Stein's two-stage sampling scheme," *Ann. Math. Stat.*, v. 24, 1953, p. 640–649.

1180[K].—J. LEFÈVRE, "Application de la théorie collective du risque à la réassurance 'Excess-Loss,'" *Skandinavisk Aktuarietidskrift*, v. 35, 1952, p. 161–187.

This paper contains a table of values of

$$B_n(u) = \int_0^\infty te^{-tu} d\phi^{(n)}(t)$$

in which $\phi(t)$ is the cumulative normal frequency function in standard units. Values are given for $n = 0, 3, 4, 6$ for $u = 0(.1)3$ to 5D.

C. C. C.

1181[K].—JACK MOSHMAN, "Critical values of the log-normal distribution," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 600–609.

The three-parameter log-normal distribution function may be written as

$$f(x) = \frac{(2\pi)^{-\frac{1}{2}}}{c(x-a)} \exp\left\{ -\frac{1}{2c^2}\left(\log \frac{x-a}{b}\right)^2 \right\}.$$

In terms of parameters a, b, and c, the mean, variance and skewness (third standard moment) of this distribution may be expressed as $\mu = b\omega^{\frac{1}{2}} + a$, $\sigma^2 = b^2\omega(\omega - 1)$, and $\alpha_3 = \pm (\omega - 1)^{\frac{1}{2}}(\omega + 2)$ where $\omega = \exp c^2$ and α_3 takes the same sign as b. The author tabulates selected critical values τ_β of the standardized log-normal variate such that $P(\tau \geqq \tau_\beta) = \beta$, where $\tau = (x - \mu)/\sigma$. Tabulations are to 3D for $\beta = .005, .01, .025, .05, .10, .90, .95, .975, .99$, and $.995$ with $\alpha_3 = 0(.05)3$. Accuracy to within one or two digits in the last decimal is claimed for all table entries and the author indicates that three point Lagrangian interpolation will give similar accuracy for intermediate values of α_3.

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1182[K].—K. R. Nair, "Tables of percentage points of the 'Studentized' extreme deviate from the sample mean," *Biometrika*, v. 39, 1952, p. 189–191.

Let $x_\nu(\nu = 1(1)n)$ be the νth ordered variate in a sample of size n taken from a normal population with unknown standard deviation σ. If an estimate s_r of σ is available with r degrees of freedom, independent of the sample, the author suggested the use of the Studentized extreme deviation $(x_n - \bar{x})/s_r$ or $(\bar{x} - x_1)/s_r$ as a test criterion for a single outlier. In a previous paper[1] he gave the lower and upper 5% and 1% points of this deviate. These tables have now been extended to cover four more percent points, namely 10, 2.5, .5 and .1%. Table 1A gives the 6 lower percentage points mentioned to 2D for $n = 3(1)9$ and $r = 10, 15, 30, \infty$. Table 1B gives the same six upper percentage points to 2D for $n = 3(1)9$ and $r = 10(1)20, 24, 30, 40, 60, 120, \infty$.

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[1] K. R. Nair, "The distribution of the extreme deviate from the sample mean and its Studentized form," *Biometrika*, v. 35, 1948, p. 118–144.

1183[K].—E. S. Pearson & H. O. Hartley, "Charts of the power function for analysis of variance tests, derived from the non-central F-distribution," *Biometrika*, v. 38, 1951, p. 112–130.

Let $u_i(i = 1, 2, \cdots, \nu)$ be ν normally distributed independent variables with unit variance and zero mean and let $a_i(i = 1, 2, \cdots, \nu)$ be ν fixed constants, then the distribution of

$$\chi'^2 = \sum_{i=1}^{\nu} (u_i + a_i)^2,$$

which is a Bessel function, is called the non-central chi-square distribution with ν degrees of freedom and $\lambda = \sum_1^\nu a_i^2$ is called the non-centrality parameter. If $\chi_1'^2$ is such a value with ν_1 degrees of freedom and χ_2^2 another independent central chi-square with ν_2, then $F' = (\chi_1'^2\nu_2)/(\chi_2^2\nu_1)$, called the non-central variance ratio, has a known distribution. Its numerical values are obtained with the help of the incomplete β functions. The probability $\beta(\lambda \mid \alpha, \nu_1, \nu_2) = Pr\ (F' > F_\alpha)$ regarded as a function of λ is the power function of the analysis of variance test with significance level α. On the basis, mainly, of Tang's table[1] eight charts are given corresponding respectively to $\nu_1 = 1(1)8$ for β at the levels $\alpha = .05$ and $\alpha = .01$. Here the non-centrality parameter $\varphi = (\lambda/(\nu_1 + 1))^{\frac{1}{2}}$ is used as the abscissa instead of λ on a linear scale. Each chart gives two families of eleven power curves corresponding to $\nu_2 = 6(1)10, 12, 15, 20, 30, 60, \infty$ for the two values of α. The use of a logarithmic scale for the ordinate β straightens the curves and expands them in the region of high power $.80 \leqq \beta \leqq .99$. The β grid, .1(.1).5(.05).7(.02).90(.01).99, and the φ grid of .2 for $\alpha = .01$ and of .1 for $\alpha = .05$ are sufficiently fine to allow interpolation by sight. The calculation of φ is shown for the one way classification into k groups with n observations in each, for double classification with one observation in each cell and for the latin square arrangement.

The use of the charts is explained for the analysis of the effect of machine variations on the standard deviation of manufactured bulk product and of the effect of personal factors introduced in routine tests.

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[1] P. C. Tang, "The power function of the analysis of variance test with tables and illustrations of their use," *Stat. Res. Memoirs*, v. 2, 1938, p. 126–157.

1184[K].—Frank Proschan, "Confidence and tolerance intervals for the normal distribution," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 550–564.

The author presents an excellent summary of confidence and tolerance intervals for the normal distribution for the various combinations of known and unknown mean and standard deviation. Let x be normally distributed with mean μ and standard deviation σ. Define

$$\bar{X} = \sum_{i=1}^{n} X_i/n, \quad s^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1).$$

Let m represent either μ or $\bar{X}$; let s.d. represent either σ or s. Then either confidence interval statements or tolerance interval statements may be made about $m \pm k$ s.d., where the value of k depends on the particular type of interval and whether or not μ and σ are known or unknown. All tables of k_i, $i = 1(1)9$ are given for $n = 2\ (1)\ 30, 40, 60, 120, \infty$, to 3D.

Given σ known, the 50% confidence interval for μ is given by $\bar{X} \pm k_1\,\sigma$, $k_1 = .6745/\sqrt{n}$; if both μ and σ are unknown, the 50% confidence interval is given by $\bar{X} \pm k_2 s$; if both μ and σ are unknown, $\overline{X_1} \pm k_3 s$ provide 50% confidence interval for the second sample mean $\overline{X_2}$, when $n_1 = n_2$.

The next k_i, $i = 4(1)7$, refer to tolerance limits. If both μ and σ are known $\mu \pm k_4\sigma$, $k_4 = .6745$ provide tolerance (probability) limits such that the proportion p of the population included by the interval is .50. In case μ and σ are unknown the factors $k_{5,a}$ in the tolerance limits $\bar{X} \pm k_{5,a}s$ are given for $a = .50, .75, .95, .99, .999$, i.e. the average p contained in $\bar{X} \pm k_{5,a}s$ will be a. If μ is unknown and σ known, tolerance limits are found by use of $\bar{X} \pm k_6\sigma$, and k_6 is tabulated for $a = .50$; for μ known, σ unknown, tolerance limits are given by $\mu \pm k_7 s$ and k_7 is tabulated for $a = .50$.

The final cases k_8 and k_9 refer to confidence statements about tolerance limits. Bowker[1] has tabulated the values of k such that the probability is γ that $\bar{X} \pm ks$ will include p or more of the population. In Bowker's tables p and γ are listed for all combinations of .75, .90, .95, .99 and .999, μ and σ unknown. The author assumes μ known and σ unknown and gives values of k_8 for $p = \gamma = .50$; and μ unknown, σ known, k_9 for $p = \gamma = .50$.

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[1] C. Eisenhart, M. W. Hastay, W. A. Wallis, editors, *Selected Techniques of Statistical Analysis*. New York and London, 1947, ch. 2 by A. H. Bowker, p. 102–107.

1185[K].—S. Rushton, "On sequential tests of the equality of variances of two normal populations with known means," *Sankhyā*, v. 12, 1952, p. 63–78.

Tables are given for sequential tests of the equality of variances in two samples from normal populations using sums of squares (a) about population means, (b) about sample means and using ranges of four and eight observations. U is the ratio of sums of squares from the two samples. Sampling is continued unless $U \leqslant U'$ (accept $\sigma_1 = \sigma_2$) or $U \geqslant U''$ (accept $\sigma_1 = \delta\sigma_2$). Tables list U' and U'' for degrees of freedom $n = 1(1)10(2)20,25,30$ and $\delta = 1.5, 2.0, 3.0$ for all combinations of α and $\beta = .01, .05, .10$. The test using ranges is based on R_n, the ratio of sums of ranges from the two samples. The tables list R_n' and R_n'' for the test as above and for the same range of parameters except that α and β are restricted to .01 and .05 and n refers to the number of ranges of 4 or 8 observations so that the sample sizes are $4n$ and $8n$.

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1186[K].—E. S. Smith, *Binomial, Normal and Poisson Probabilities.* Published by the author, Box 224C, RD2, Bel Air, Md., 1953. 71 p., 21.6 × 27.9 cm. $2.50.

This is a small set of tables and charts, centering on the cumulative binomial distribution, with extended discussion of actual (some novel) as well as alternative methods of computation. Many of the tables shown are obtainable from larger tables constructed by others (referred to in the text). Tables are:

(1) Maximum Poisson probabilities $p_i(x, a)$ for various a (x, of course, $= 0, a, a - 1$), a monotone function falling from .999001 (at $x = 0$) for $a = .001$ to .039861 for $a = 100$ to 6D

(2)	$n!$ and $\log n!$,	$n = 0(1)200$	10S and 10D respectively
(3)	$\log n$,	$n = 1(.01)10$	10D
(4)	C_x^n and $\log C_x^n$,	$n = 1(1)50$	5S and 5D respectively
(5)	e^{-x},	$x = 0(.001)1(1)100$	10D
(6)	$B(c,n,p)$,	$n = 1(1)20;\ p = .01(.01).5$	5D

where $B(c,n,p) = \sum_{x=c}^{n} C_x^n p^x (1 - p)^{n-x}$

(7) Normal (Gaussian) integrals $\int_0^t \phi(y)dy$, density functions $\phi(t)$, second derivatives $\phi^{(2)}(t)$, $t = 0(.01)4$, to 5D.

(8) $p(c,a)$, $c = 1(1)22$; $a = .001(.001).01(.01).1(.1)2(1)10$ and $\frac{c}{a} = .1(.1)2.2$; $a = 10(10)100$ to 5D,

where $p(c,a) = \sum_{x=c}^{\infty} e^{-a}a^x/x!$

The cumulative binomial probabilities are obtained (a) directly, (b) from the Poisson cumulatives (singly or the two-term Gram-Charlier Type B), (c) from the normal (singly or the two-term Gram-Charlier Type A, with or without remainder modifications) depending on the ranges of the binomial parameters. There are a number of useful charts, not, to my knowledge, to be found elsewhere. These include (1) the probability of n successes in n trials, with constant probability of success in single trial, (2) the expected number (np) of binomial successes, (3) (most useful of all) a set of charts, some expressed in terms of correction factors, comparing the normal, binomial, and Poisson probabilities for various ranges of the parameters.

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1187[K].—H. Theil, "On the time shape of economic microvariables and the Munich business test," Inst. International Stat., *Revue*, v. 20, 1952, p. 105–120.

This paper contains a table giving the frequency distributions of the difference between two independent random variables each obeying Poisson distributions with means m and μ respectively. Values are given to 3D for $m,\mu = \frac{1}{2}, 1, 2, 4$.

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1188[K].—H. Uranisi, "The distribution of statistics drawn from the Gram-Charlier Type A population," *Bull. Math. Stat.*, v. 4, 1950, p. 1–14.

From the expansion of a frequency function in Gram-Charlier Type A series, the author obtains early terms of a similar expansion of the distribution of the t-statistic for a sample of n. For both the one-tailed and two-tailed cases, four coefficients of this series are tabled to 6D for $n = 5, 10, 15, 21$ for values of the argument $u = (1 + t^2/[n-1])^{-1} = .05, .1(.1).9, .95, 1$. With the aid of these tables it is possible to estimate the tail probabilities for given values of β_1, β_2, β_3, and β_4, and this is done for several examples. The results are not entirely comparable with those of Gayen,[1] in that the latter employed the Edgeworth series.

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[1] A. K. Gayen, "The distribution of 'Student's' t in random samples of any size drawn from non-normal universes," *Biometrika*, v. 36, 1949, p. 353–369.

1189[K].—J. Westenberg, "A tabulation of the median test with comments and corrections to previous papers," K. Ned. Akad. v. Wetensch., *Proc.*, v. 55, s.A, 1952, p. 10–15.

Let $X_1, \cdots, X_{N_1}$ and $Y_1, \cdots, Y_{N_2}$ be samples of N_1, and N_2 observations drawn at random from populations having continuous distribution functions $F_1(x)$ and $F_2(y)$ respectively. Let δ be the number of observations belonging to one of the samples that lies between the median of that sample and the median of the combined sample. The hypothesis H_0 that $F_1 \equiv F_2$ is rejected

when δ exceeds a critical value δ_0. Tables of δ_0 to 1D are given for significance levels .001, .005, .01, (.01), .05 when considering two-sided alternatives to H_0 and .0005, .0025, .005, .01, .015, .02, .025 when considering one-sided alternatives to H_0 for $N_1, N_2 = 6, 10, 20, 50, 100, 200, 500, 1000, 2000$.

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1190[K].—J. W. WHITFIELD, "The distribution of total rank value for one particular object in m rankings of n objects," *British Jn. of Stat. Psychology*, v. 6, 1953, p. 35–40.

The problem considered is, essentially, that of the distribution of the sum of m independent random variables, each with uniform distribution on the integers $1, 2, \cdots, n$. The distributions, of course, are symmetric about $\frac{1}{2}m(n + 1)$. Three pages of tables give the lower halves of the cumulative distribution functions to 5D for $m = 2\ (1)\ 8$ and $n = 3\ (1)\ 8$.

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1191[L].—C. DOMB, "Tables of functions occurring in the diffraction of electromagnetic waves by the earth," *Advances in Physics*, v. 2, 1953, p. 96–106.

The tables given in the paper are related to the Airy integral

$$\mathrm{Ai}(z) = \pi^{-1} \int_0^\infty \cos\left(\tfrac{1}{3}t^3 + zt\right)dt$$

in the complex plane.

The numbers $-a_n$ being the zeros of $\mathrm{Ai}(z)$, the author puts

$$f_n(y) = \exp\left(\lambda_n + i\mu_n\right) = \frac{\mathrm{Ai}[-a_n + y\exp(\pi i/3)]}{\mathrm{Ai}'(-a_n)\exp(\pi i/3)};$$

the numbers b_n being roots of the equation

$$\mathrm{Ai}(z) = t\mathrm{Ai}'(z)\exp(-5\pi i/12),$$

in which t is a parameter, he also puts

$$\xi_n + i\eta_n = b_n \exp(\pi i/6)$$

and

$$\exp(\gamma_n + i\delta_n) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\exp(-5\pi i/12)[\mathrm{Ai}'(b_n)]^{-1} \times [1 - t^2 b_n \exp(-5\pi i/12)]^{-\frac{1}{2}}.$$

Tables of the Airy integral for real argument have been reviewed in *MTAC*, v. 2, p. 302–305, RMT **413**, and for complex argument, in v. 2, p. 309, RMT **420**.

The author tabulated the functions $f_n(y)$ for $n = 1(1)5$ in 1942, under the guidance of J. C. P. MILLER. These tables were subsequently checked, corrected, and sub-tabulated by the Mathematics Division of the National

Physical Laboratory of Great Britain. Photostatic copies are available from H. M. Nautical Almanac Office, Great Britain. The computations of Admiralty Computing Service are described in *MTAC*, v. 2, p. 35, RMT **260**.

Table 1 of the present paper gives 3D values of λ_n and μ_n for $n = 1(1)5$, $y = 0(.2)3(1)10$ and 3D values of λ_n and $\mu_n + \frac{3}{4}y^{\frac{1}{2}}$ for $n = 1(1)5$ and $y = 10(10)100$.

Table 2 gives 3D values of ξ_n and η_n for $n = 1(1)5$, $t = 0(.1)1$, $t^{-1} = 1(-.1)0$.

Table 3 gives 3D values of γ_n and δ_n for $n = 1(1)5$ and for t ranging from 0 to ∞; the number of selected values of t varies with n.

There is a brief description of the computations, and an indication of the application of the functions tabulated here.

A. E.

1192[L].—M. Mashiko, *Tables of Generalized Exponential-, Sine-, and Cosine Integrals* $\mathrm{Ei}(x + iy)$, $\mathrm{Si}(x + iy)$, $\mathrm{Ci}(x + iy)$. Numerical Computation Bureau, Tokyo, Japan, Report No. 7, March 1953, 43 p.

Let

$$I(z) = \int_z^\infty t^{-1}e^{-t}dt.$$

For $z = \xi e^{i\alpha}$ $(0 \leq \xi \leq 5)$ put

$$I(\xi e^{i\alpha}) = C_\alpha(\xi) - iS_\alpha(\xi).$$

For $z = \dfrac{1}{\eta} e^{i\alpha} (\eta \leq 0.2)$ put

$$I\left(\frac{1}{\eta}e^{i\alpha}\right) = \frac{e^{-z}}{z} A_\alpha(\eta) \exp(i\Phi_\alpha(\eta)).$$

The report contains values of

(a) $C_\alpha(\xi) + \log \xi$ and $S_\alpha(\xi)$ to six decimal places with second differences for $\xi = 0(.05)5.00$, $\alpha = 0°(2°)60°(1°)90°$.

(b) $A_\alpha(\eta)$ to six decimal places, $\Phi_\alpha(\eta)$ to five decimal places, each with second differences, for the above range of α and $\eta = 0(.01).20$. In defining the ranges, the reversals of the inequality signs are presumably misprints.

The present table is therefore concerned with the exponential integral for complex argument, not the generalized exponential-integral which has been tabulated by the Harvard University Computation Laboratory [*MTAC*, v. 4, p. 92–93]. It breaks new ground in covering, for complex arguments in polar form, the whole first quadrant.

No statement is made as to the accuracy of the table. Spotchecking a few values against the forthcoming NBS *Tables of Exponential Integrals for Complex Arguments* (cartesian form) did not reveal any discrepancy.

Everett's interpolation formula involving second differences gives the maximum attainable accuracy in both directions. In order to facilitate interpolation, a four decimal place table of Everett second-order interpolation coefficients for arguments at intervals of .01 is also given.

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1193[L].—NBSCL, "Struve function of order three-halves," NBS, *Jn. Research*, v. 50, 1953, p. 21–29.

Tables of

$$h_{\frac{3}{2}}(x) = \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} H_{\frac{3}{2}}(x)$$

$$= 1 + \frac{2}{x^2} - \frac{2}{x}\left(\sin x + \frac{\cos x}{x}\right),$$

to 10D, with second central differences, modified when necessary (modification being indicated by a letter C placed after the entry); $x = 0(.02)15$. "The values are expected to be correct to within one unit of the last place."

The entire computation of this table was done on the SEAC, under the supervision of ETHEL MARDEN, by KATHRYN CHRISTOPH, ANNE FUTTERMAN, RENEE JASPER, SALLY TSINGOU, and BERNARD URBAN.

The table is also available on IBM cards.

A. E.

1194[L].—HERBERT E. SALZER, RUTH ZUCKER, & RUTH CAPUANO, "Table of the zeros and weight factors of the first twenty Hermite polynomials," *Jn. Research*, NBS, v. 48, 1952, p. 111–116.

The definition of Hermite polynomials used in this paper is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}).$$

$x_i^{(n)}$ is the i-th positive zero of $H_n(x)$,

$$\alpha_i^{(n)} = \pi^{\frac{1}{2}} 2^{n+1} n! [H_n'(x_i^{(n)})]^{-2}$$

is the corresponding Christoffel number, and

$$\beta_i^{(n)} = \alpha_i^{(n)} \exp[(x_i^{(n)})^2].$$

The present paper gives 15D values of $x_i^{(n)}$ and 13S values of $\alpha_i^{(n)}$ and $\beta_i^{(n)}$ for $n = 1(1)20$, $i = 1(1)n$. A list of references (28 items) is appended.

Other tables of zeros of Hermite polynomials are referred to in *MTAC*, v. 1, p. 152, RMT **131**; v. 3, p. 26, RMT **466**; v. 3, p. 416, RMT **619**; v. 3, p. 473, RMT **641**; v. 6, p. 232, RMT **1034**. The present tables were compared with those of REIZ [RMT **466**], GREENWOOD & MILLER [RMT **619**] and KOPAL [RMT **641**], and with the HARVARD tables [RMT **1034**].

A. E.

1195[L].—K. M. SIEGEL, J. W. CRISPIN, R. E. KLEINMAN & H. E. HUNTER, "Note on the zeros of $(dP_{m_i}^{(1)}(x)/dx)|_{x=x_0}$," *Jn. Math. Phys.*, v. 32, 1953, p. 193–196.

The authors apply the identical technique of a previous paper to obtain approximate values m_i such that $dP^{(1)}{}_{m_i}(x_0)/dx = 0$ and values of

$$\int_{x_0}^{1} x_0[P_{m_i}^{(1)}(x)]^2\, dx$$

[see *MTAC*, v. 7, p. 183]. Again, the theory is demonstrated for $x_0 =$

cos 165°. For this case, the first 15 approximate values of m_i and corresponding values of the integral are tabulated.

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1196[L].—N. B. SLATER, "Gaseous unimolecular reactions: theory of the effects of pressure and of vibrational degeneracy," Roy. Soc. London, *Philos. Trans.*, s.A, v. 246, 1953, p. 57–80.

Table 3 (p. 71) gives values to 2 or 3S of

$$I_n(\theta) = [\Gamma(m+1)]^{-1} \int_0^\infty x^m e^{-x}(1 + \theta^{-1}x^m)^{-1}dx \qquad m = (n + 1)/2$$

for $n = 3(2)13$ and some of the values $\log_{10}\theta = - 2(1)8$.

Table 4 (p. 71) gives 2 or 3S values for $n = 3(2)13$ of θ_5 and θ_{50} such that $I_n(\theta_5) = .95$, $I_n(\theta_{50}) = .50$, and also θ_5/θ_{50}.

A. E.

1197[L].—R. C. T. SMITH, "Conduction of heat in the semi-infinite solid, with a short table of an important integral," *Australian Jn. Phys.*, v. 6, 1953, p. 127–130.

Table 1 (p. 128–129) gives 5D values of

$$\int_0^U (1 + u^2)^{-1} \exp[-\alpha(1 + u^2)]du$$

for $\alpha = .1(.1)2$ and $U = .1(.1)2, 2.5, 3, \infty$, and also for $\alpha = 2.5, 3, 4, 5$ and a shorter range of U.

A. E.

1198[L].—MICHAEL TIKSON, "Tabulation of an integral arising in the theory of cooperative phenomena," NBS, *Jn. Research*, v. 50, 1953, p. 177–178.

Table 1. Values of the coefficients c_{2m} in the expansion

$$[I_0(x)]^3 = \sum_0^\infty c_{2m}x^{2m}$$

for $m = 0(1)20$. Here $I_0(x)$ is the modified Bessel function of order zero.

Table 2. Values of

$$I(b) = \pi^{-3} \int_0^\pi \int_0^\pi \int_0^\pi [3b - (\cos x + \cos y + \cos z)]^{-1}dx\, dy\, dz$$

$$= \sum_0^\infty (2m)!\, c_{2m}\, (3b)^{-2m-1}$$

to 5D for $b^{-1} = .01(.01)1$. For $b^{-1} \leq .8$, the expansion in powers of b^{-1} was used to compute $I(b)$, at most 21 terms of this expansion being required. The remaining values of $I(b)$ were obtained by numerical integration.

A. E.

1199[L,V].—C. Truesdell, "Precise theory of the absorption and dispersion of forced plane infinitesimal waves according to the Navier-Stokes equations," *Jn. Rational Mech. and Analysis*, v. 2, 1953, p. 643–741. Tables (p. 723–734) computed by Harrison Hancock. Graphs (p. 735–741).

These tables give results on the calculation of the propagation of plane infinitesimal pressure waves (sound waves) in a uniform fluid governed by the Navier-Stokes equations. The fluid is characterized by the viscosity coefficients μ and λ, the coefficient of heat conduction κ, and the specific heats c_p, and c_v. The tables are divided according to the two parameters, the ratio of specific heats

$$\gamma = c_p/c_v$$

and the thermoviscous number Y,

$$Y = \frac{\kappa}{(\lambda + 2\mu)c_p}.$$

Tables given are for (γ, Y) of the following values $(1, Y)$ (piezotropic fluids); (1.10, .5), (1.10, .8), (1.25, .25), (1.25, .5), (1.25, .8), (1.40, .25), (1.40, .5), (1.40, .8), $\left(\frac{5}{3}, .25\right)$, $\left(\frac{5}{3}, .5\right)$, $\left(\frac{5}{3}, .6\right)$, $\left(\frac{5}{3}, .8\right)$, $\left(\frac{5}{3}, 1.05\right)$, $\left(\frac{5}{3}, 1.25\right)$, (1.8, .3), (1.8, .5), (2, .3), (2, .5), (2.2, .3), (2.2, .5) (weak and moderate conductors); (1.10, 20), (1.10, 30), (1.10, 40), (1.15, 10), (1.15, 20), (1.15, 25), (1.15, 30), (1.15, 35), (1.15, 40), (1.20, 20), (1.20, 30), (1.20, 40), (1.50, 50) (strong conductors). In each table, the various dimensionless quantities characterizing the speed, the absorption and the dispersion of a plane sound wave of circular frequency ω are listed against the argument X, the frequency number

$$X = \frac{(\lambda + 2\mu)}{\gamma p}$$

where p the undisturbed pressure of the fluid. The ranges of argument is $X = .1(.2).7, 1(.5)5$.

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1200[L].—G. Zartarian & H. M. Voss, "On the evaluation of the function $f_\lambda(M, \omega)$," *Jn. Aeron. Sciences*, v. 20, 1953, p. 781–782.

9D table of the coefficients

$$a_{2n}(M) = \sum_{j=0}^{n} \frac{(2M)^{-2j}}{(j!)^2[2(n-j)]!},$$

$$b_{2n}(M) = \sum_{j=0}^{n} \frac{(2M)^{-2j}}{(j!)^2[2(n-j)+1]!}$$

for $n = 0(1)6$, $M = 5/4, 10/7, 3/2, 5/3, 2, 5/2$. These coefficients occur in

the expansion

$$\int_0^1 e^{-iu\omega} J_0(\omega u/M) u^\lambda du$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{a_{2n}(M)}{\lambda + 2n + 1} - i\omega \frac{b_{2n}(M)}{\lambda + 2n + 2} \right) \omega^{2n},$$

and may be used for computing the integral.

A. E.

1201[Q].—H. Wood, (a) Kepler's problem, Roy. Soc. of New South Wales, Jn. and Proc., v. 83, 1950, p. 150–163; (b) Kepler's problem—the parabolic case, v. 83, 1950, p. 181–194; (c) Tables for nearly parabolic elliptic motion, v. 84, 1951, p. 134–150; (d) Tables for hyperbolic motion, v. 84, 1951, p. 151–164; (e) Five figure tables for the calculation of ephemerides in parabolic and nearly parabolic motion, Sydney Observatory Papers No. 16, 1951. (*a*), (*b*), (*c*) and (*d*) are also Sydney Observatory Papers Nos. 10, 11, 14 and 15 respectively

The first four papers give the theory and seven-figure tables which are especially applicable to the problem of calculating the ephemerides of comets in parabolic or nearly parabolic orbits about the sun. The fifth paper gives five-figure tables which are sufficiently accurate for finding purposes. Some of these tabulations may be of more general mathematical interest.

The two body problem was solved (kinematically) by Kepler with the enunciation of his three laws of planetary motion. The problem of finding the coordinates of a planet, or comet, in the plane of its orbit in unperturbed motion is Kepler's problem. If M, E and v are the mean, eccentric and true anomalies, respectively, and e the eccentricity, then: $M = E - e \sin E$, where $\tan E/2 = ((1-e)/(1+e))^{\frac{1}{2}} \tan v/2$. This equation is Kepler's equation; the necessity for its frequent, accurate solution, and the difficulties, both numerical and analytical which it presents, have kept alive interest in the equation for the past 300 years. In general, e is quite small for the orbits of planets, asteroids and most binary stars and there are numerous satisfactory methods of solution. Cometary orbits, on the other hand, are very often parabolic, or nearly so, and are sometimes even hyperbolic because of planetary perturbations, or fictitiously hyperbolic because of observational errors. If e is close to unity and E small, special methods must be devised to solve Kepler's equation accurately. Wood writes the equation as follows:

$$D = 12k\,(1+e)^{\frac{1}{2}} q^{-\frac{3}{2}} t = 12\mu + \mu^3(1+\epsilon)6 \left(\frac{\sin^{-1}\epsilon^{\frac{1}{2}}\mu - \epsilon^{\frac{1}{2}}\mu}{\epsilon^{\frac{3}{2}}\mu^3} \right),$$

where k is the Gaussian gravitational constant, q the perihelion distance, t the time ($t = 0$ at perihelion), $\epsilon = (1-e)/(1+e)$ and $\mu = y_0/q$ where x_0 and y_0 are the rectangular coordinates in the orbital plane with the x_0-axis directed towards perihelion.

For $\epsilon = 0$ ($e = 1$) we have the parabolic case and

$$D = 12\mu = \mu^3, \quad \text{where } \mu = 2 \tan v/2.$$

Table 1 in (*b*) gives μ to 7D for $D = 0(0.1)100$; to 6D for $D = 100(1)1000$. For $D > 1000$ use $\mu = D^{\frac{1}{3}} - 4/D^{\frac{1}{3}} + R$. Table 2 in (*b*) gives R to 6D for

$4/D^{1/3} = 0(.01)0.41$. Previous tables are useful only up to $D = 88$ and Wood states that there have been 36 comets observed outside this range of D, with the probability that the future will see an increasingly higher percentage of such observations.

The above tables are also useful in the nearly parabolic case. For $e \neq 1$, D is redefined as:

$$D = 12k(1 + e)^{1/2}q^{-3/2}ct = 12c\sigma + c^3\sigma^3,$$

and

$$\mu = \frac{c\sigma}{c}\{J - hK + R\},$$

where c^2 and h are certain power series in ϵ, and J and K are certain power series in $\epsilon\sigma^2$. The coefficients of the last terms used in the K and h series have been adjusted so that R is negligible in the seventh decimal place. σ is defined by the top equation and $c\sigma$ can be evaluated from Table 1 in (*b*). c is tabulated to 7D and h to 5D for $\epsilon = 0(.001)0.100$ in Table 2 in (*c*) (ellipse) and for $\alpha(= -\epsilon) = 0(.001)0.100$ in Table 2 in (*d*) (hyperbola). J and K are given to 7D for $\epsilon^{1/2}\sigma = 0(.001)0.600$ in Table 3 in (*c*) and similarly for $\alpha^{1/2}\sigma$ in Table 3 in (*d*).

Other useful quantities tabulated by Wood are of the form:

$$A = 6\left\{\frac{\sin^{-1}w - w}{w^3}\right\}, \quad I = 6\left\{\frac{1-\sqrt{1-w^2}}{w^2\sqrt{1-w^2}}\right\}, \quad \text{and } N = 2\left\{\frac{1-\sqrt{1-w^2}}{w^2}\right\},$$

where $w = \epsilon^{1/2}\mu$ for the ellipse and:

$$A = 6\left\{\frac{w - \sinh^{-1}w}{w^3}\right\}, \quad I = 6\left\{\frac{\sqrt{1+w^2}-1}{w^2\sqrt{1+w^2}}\right\}, \quad \text{and } N = 2\left\{\frac{\sqrt{1+w^2}-1}{w^2}\right\},$$

where $w = \alpha^{1/2}\mu$ for the hyperbola. If we define D_1 as:

$$D_1 = (1 + e)\,6\mu + A\mu^3, \quad \text{then } \frac{dD_1}{d\mu} = 6(1 + e) + \mu^2 I.$$

A and I are useful in the iterative computation of μ and of velocities. The other rectangular coordinate in the orbital plane is $\lambda = x_0/q = 1 - \mu^2N/2$ $(1 + e)$. This becomes $\lambda = 1 - \mu^2/4$ for the parabolic case. Appropriate formulae are given to calculate the equatorial heliocentric coordinates x, y and z, from N and μ.

A and N are tabulated to $7D$ and I to 4D for $\epsilon^{1/2}\mu = 0(.001)0.600$ in Table 1 of (*c*), and similarly for $\alpha^{1/2}\mu = 0(.001)0.600$ in Table 1 of (*d*).

Paper (*e*) gives 5D tables applicable to the three cases considered. There has been some slight rearrangement of the formulae and tabulated quantities but the details are probably not of sufficient general interest and will not be given here. The author states that, in general, the tables have been calculated to two extra decimal places. A very useful bibliography will be found in paper (*a*).

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to errata in RMT **1165** and UMT **184.**

236.—Tosio Kitagawa, *Tables of Poisson Distributions*. Baifukan, Tokyo, 1952.

The above tables have been compared with an unpublished table for $m = 1(1)10$. One misprint and four rounding errors appear as follows:

m	*x*	*for*	*read*
7.00	4	.0902262	.0912262
9.00	15	.0194306	.0194307
9.00	24	.0000158	.0000159
10.00	11	.1137363	.1137364
10.00	15	.0347180	.0347181

D. Teichroew

NBSINA

237.—M. Kraitchik, *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924.

This work contains (p. 131–191) a table of residue-indices of 2 modulo P for $P < 300000$ as described in UMT 184.

The following errata are new [for other errata see MTE **107**, *MTAC*, v. 2, p. 313].

P	*for*	*read*
101681	10	2
102329	8	4
103183	1	2
103669	1	3
104239	1	2
104383	1	2
104527	1	2
106273	1	2
106663	1	2
106963	1	3
107581	3	1
107941	1	3
108571	1	3
109433	4	8

D. H. L.

238.—National Bureau of Standards. *Table of Bessel Functions of Fractional Order*, v. 1, New York, Columbia University Press, 1948.

Function $J_{-\frac{1}{4}}(x)$

x	Column	*For*	*Read*
20.84	δ^2	12305	12299

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UNPUBLISHED MATHEMATICAL TABLES

181[E].—National Bureau of Standards, *Tables of* sinh x *and* cosh x. Available on IBM cards at NBSCL.

A table of sinh x, cosh x for $x = 2(.001)10$ to 9S has been prepared on SEAC. This is an extension of the material available in National Bureau of Standards Applied Math. Ser. No. 36. Among those who contributed to the preparation of this table were W. F. Cahill and S. B. Prusch.

J. T.

182[E].—National Bureau of Standards, *Table of* $\exp(-x)$. Available on IBM cards at NBSCL.

A table of $\exp(-x)$, $x = 2.5(.001)10$, to 20D has been prepared on SEAC. This is an extension of the material in NBS Applied Math. Ser. No. 14 which gives $\exp(-x)$ for $x = -2.5(.0001)2.5(.001)5(.01)10$ to at least 12D.

Among those who contributed to the preparation of this table were R. M. Davis, E. C. Marden, M. Paulsen and S. B. Prusch.

J. T.

183[F].—A. Gloden, *Factorization of* $2N^4 + 1$, $N \leq 1000$. 17 typewritten pages deposited in the UMT File.

Factorizations are complete up to $N = 100$ except for $N = 73$. They are mostly incomplete beyond $N = 500$.

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184[F].—F. Gruenberger, *Table of residue-indices of* 2 *for prime moduli between* 100000 *and* 110000. One sheet tabulated from punched cards deposited in the UMT File. Other copies available from the author.

This table is the beginning of a recalculation of an unreliable table of Kraitchik[1] for $p < 300000$. It gives for each p between the limits indicated the integer $\nu = (p - 1)/x$ where x is the "exponent" of 2, that is, the least positive integer n for which $2^n - 1$ is divisible by p. New errors uncovered in Kraitchik's table by collation with the present table are given in MTE **237** in this issue.

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[1] M. Kraitchik, *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924, p. 131–191.

185[F].—R. J. Porter, *Additional Tables of Irregular Negative Determinants of Exponent* $3n$. Typewritten manuscript on deposit in the UMT File.

This is an extension of UMT **155** [*MTAC*, v. 7, p. 34] to determinants $-D$ with $50000 \leq D < 80000$.

In this range there are three D's with exponent of irregularity equal to 6, namely

$$D = 55555,\ 67899,\ 70244.$$

As in the earlier table, there are many more D's with exponent 9 in fact the following 11.

52731, 54675, 56403, 58563, 60075, 64395, 70227, 70956, 75411, 76059, 77571.

All 1169 other D's have exponent 3.

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186[L].—C.-E. FRÖBERG & P. RABINOWITZ, *Tables of Coulomb Wave Functions* NBS Report, 12 January 1954. Deposited in the UMT FILE.

The first volume of a table of *Coulomb Wave Functions* was issued in the NBS Applied Math. Ser. in 1952 (see RMT **1091**); it is devoted to the regular solution and enables values to about 7 figures to be obtained in the range $L = 0(1)21$, $-6 \leq \eta < 6$, $0 \leq \rho \leq 5$. A second volume will be issued in the same series; it will be concerned with the regular and irregular solution for $L = 0$ and in the range $0 \leq \eta \leq 10$, $0 \leq \rho \leq 10$. This report, of which a limited number of copies are available, is concerned with a skeleton table for both the regular and irregular solutions. From this table it is possible to obtain both the functions to about 6–7 figures in the range $L = 0(1)21$, $1 \leq \eta \leq 10$, $1 \leq \rho \leq 10$, using standard techniques of interpolation and the recurrence relations for the functions.

Although this table itself will be of considerable use to physicists who are prepared to do a fairly heavy interpolation job, it is best to regard it as the result of proving in a SEAC code for the whole range $L = 0(1)25$, $0 \leq \eta \leq 100$, $\frac{1}{2} \leq \rho \leq 100$. From this code, by insertion of suitable parameters, it is possible to obtain a detailed table of the functions in any part of the range mentioned. This is prepared, in the first place on magnetic wire, and from this IBM cards can be punched, or a Flexowriter manuscript obtained. At present it is not thought likely that more detailed tables will be printed, but consideration can be given to the preparation of decks or manuscripts to satisfy individual or organizational needs.

A preliminary report on the method of computation, including some new integral representations of the functions, was given as Paper No. 9, at the NBS Tables Conference on May 15, 1952.

J. T.

187[L].—NATIONAL BUREAU OF STANDARDS, *Table of the Sievert Integral.*

The Sievert integral is defined by

$$y(\theta, x) = \int_0^\theta \exp(-x \sec t)dt$$

(see **Q.19**, *MTAC*, v. 2, p. 196). For $\theta = \frac{1}{2}\pi$, $y(\theta, x)$ coincides with $Ki_1(x)$ which has been tabulated by BICKLEY & NAYLER[1].

This function has been tabulated using SEAC, the computations being planned by H. E. SALZER. Among those who contributed to the preparation were R. B. JASPER, P. J. O'HARA, M. PAULSEN, and W. R. SODERQUIST. The quadratures were checked by comparison with the Bickley-Nayler table. There is now available, on IBM cards, values to 9S of $y(\theta, x)$ for

$$\theta = 0(1°)90°, \quad x = 0(.01)2(.02)5(.05)10.$$

A table to about 5S for $x = 0(.5)10$ and the same values of θ by SIDNEY JOHNSTON is referred to in UMT **103**, *MTAC* v. 4, p. 163.

J. T.

[1] W. G. BICKLEY & J. NAYLER, *Phil. Mag.*, s. 7, v. 20, 1935, p. 343–347.

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 415 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

THE DEVELOPMENT OF A.P.E.(X).C.

1. Historical.—It is the purpose of this paper to give a technical description of the all purpose electronic digital computer developed at the Birkbeck College computation laboratory. Before embarking on this, however, it seems appropriate to mention certain historical details which put the present machine into perspective.

In 1946 the present author was engaged in the development of a special purpose relay digital calculator for the automatic evaluation of three dimensional Fourier sums[1] of the type:

$$\rho(x, y, z) = \sum_{-H}^{+H} \sum_{-K}^{+K} \sum_{-L}^{+L} |F_{hkl}| \cos\left\{ 2\pi\left(\frac{hx}{a} + \frac{ky}{b} + \frac{lz}{c}\right) - \alpha_{hkl} \right\}$$

where the F and α are given in digital form and are available from a punched tape.

The complete design for this machine involved a magnetic *disc* store for 256 words of 16 bits, and also a matrix function table (one-many, many-one) for producing the cosine function.

Although this machine was to be relatively compact (600 relays and 100 vacuum tubes) it did not satisfy aesthetically since the input had to be in binary form and output resulted in a like manner.

Fortunately, at this point, the author was invited to address the American Society for X-ray and Electron Diffraction, and in the course of this visit to the U.S.A. was able to make contact with the pioneer work which was being carried out at several centres.

Through the generosity of the Rockefeller Foundation a period of study at the Institute for Advanced Study followed, during which plans were completed for an all purpose relay computer using roughly the same equipment as was already available for the earlier project.

Thus, by the end of 1947, the A.R.C. (Automatic Relay Computer) was completed. This machine, which is still available for use in this laboratory, employs 800 high-speed relays to form a parallel operation single address code computer.[2] The storage organ, which was originally on a nickel plated brass drum,[3] had a capacity of 256 words each of 21 bit length. This was later (vide infra) replaced by an electro-mechanical store for 50 words of the same length and the control modified to a pluggable unit for 600 instructions.

The reason for this seeming *volta face* lay in the use to which the machine was eventually put, which was essentially the Fourier series evaluation mentioned earlier. It has proved far more efficient to have a pluggable sequence machine when this is used only intermittently (not more than three or four times during a year) and then only by research students.

Input to the A.R.C. is *via* teletype tape or push buttons and the output is by means of a teleprinter. The code[2] is as follows:

	No.	Symbol	Description
	0	T to M	Fill memory from input tape.
	1	T to Ti	Start tape moving to Ti and proceed with next order.
	2	Tij to M	Read material between Ti and Tj into first $(j - i)$ positions in memory.
(0)	3	Mij to T	Punch contents of memory position i to j into tape.
(0)	4	C to $M(x)$	Shift control to order located at $M(x)$.
	5	Cc to $M(x)$	If number in accumulator $\geqslant 0$ shift control as in 4.
	6	$l(n)$	Shift contents of accumulator and register (A and R) n places to left. $A(0)\ A(1) - A(20)$; $R(0)\ R(1) - R(20)$ becomes $A(0), A(2) - A(20), R(0); R(1) - R(20), A(1)$, etc.
	7	$r(n)$	Shift contents of A n places to right. $A(0), A(1) - A(20)$ becomes $A(0), A(0), A(1) - A(19)$, etc.
	8	$+M$ to cA	Clear A and add contents of $M(x)$ into it.
	9	$+\|M\|$ to cA	
	10	$-M$ to cA	
	11	$-\|M\|$ to cA	
	12	$+M$ to A	Add contents of $M(x)$ into present contents of A.
	13	$+\|M\|$ to A	
	14	$-M$ to A	
	15	$-\|M\|$ to A	
	16	MR to cA	Clear A. Multiply $M(x)$ by R, place 1st 20 digits and sign in A after adding unity to 21st. Leave last 20 digits in R.
	17	MR to cA (n.r.o.)	As in 16 but without round off.
	18	$A \div M$ to cR	Divide A by $M(x)$, place quotient in R with last digit unity. Leave remainder in A.
	19	M to cR	
	20	R to cA	
	21	R to M	
	22	A to M	
(0)	23	A_L to M	Substitute digits 1–8 of A in order located at $M(x)$.
	24	A to cR	
	25	E	Stop and emit end of calculation signal.

Notice that those instructions marked (0) are no longer available since the alteration in control structure, and that the original code-word was constructed as follows:

Digit	1 to 8	9 to 15	16 to 20
	Memory location (x)	Sequence	Order Code

The zeroth digit was always zero in an instruction and the position marked "sequence" was used in orders (2) and (3) to specify $(j - i)$.

The speeds of the various arithmetic operations are:

$+, -.$	20 m. sec.
$\times, \div$	1 sec.
$l(n)$ and $r(n)$	$n \times 20$ m. sec.

In its original form (i.e. on the magnetic drum) transfers to and from M required 20 m. sec. (max); in the current electro-mechanical device a maximum of 250 m. sec. is needed, but this can be reduced by optimum programming, to 20 m. sec.

A partial reason for the removal of the magnetic drum store from A.R.C. was a desire to make this unit a part of a small all-electronic calculator. This machine, later called S.E.C.[4] (simple electronic calculator) was never brought to a state of useful operation, but served as a test bed for the designs of the A.P.E.(X).C. machines which followed. The interesting point about S.E.C. lies in the small number of vacuum tubes used in its construction (230). The code was rudimentary and the input/output took the form of switches and neon lamps. A major change in logic was the adoption of a two address code[5] of the type: "Perform operation A on number in position (x) and go to position (y) for the next instruction."

This extension removes the chief speed limitation which is inherent in a magnetic drum store and in favourable cases, increases the speed of operation by a factor of 16–32.

2. The A.P.E.(X).C. [All Purpose Electronic (X-initial of sponsoring agency) Computer].—At a relatively early stage in the actual construction of S.E.C., it was felt that sufficient experience had been gained to justify the development of a full-scale machine. A set of standard circuits had been devised[6,7,8] and the problems of drum storage were considered solved. Essential details are as follows:

Drum:	Speed 3,800 R.P.M.		
	Digit packing 50 bits/inch.		
	Track packing 32 tracks/inch.		
	Capacity 512 words of 32 bits.		
	Repetition rate 30 K.C./sec.		
Tubes:	Three major types	6.J.6.	
		6.A.L.5.	
		6.A.G.5.	(only 6 of these).

Standard Circuits.[8]

Binary cell.
Binary counting cell.
Two input gate.
Diode buffer.
Amplifier inverter.

No germanium diodes were used, as tests had not shown them to have the reliability which was considered needful (in 1949).

The first machine of A.P.E.(X).C. type was constructed for the British Rayon Research Association; no attempt was made to eliminate redundancies in the design but the resulting machine, shown in Fig. 1, still had less than 500 tubes (no germanium or other semi-conductor diodes). In many

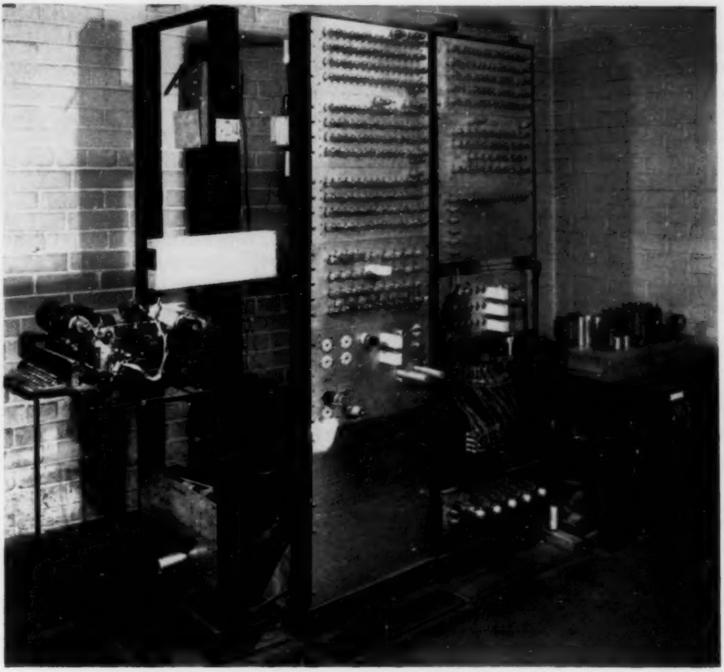

Fig. 1. The A.P.E.(R).C.

ways this machine was still a prototype, the input-output system was not fully decided and certain problems connected with track selection had still to be solved. Nevertheless from its completion in June 1952, with only 64 storage positions and teletype input/output, the machine proved reliable and useful.

By January 1953 the complete storage for 512 words was available and punched card input/output had been incorporated. From that time, until its delivery to the sponsoring agency in July 1953 the machine was in continuous use in this laboratory, and it is interesting to note that in the first 500 hours of operational use only 30 hours of faulty operation were noted (94% good operation). Moreover, no regular maintenance was done during this period:

Amongst the problems solved on the above machine (A.P.E.(R).C.) are:

Octal-Decimal Conversion table 1000 values and differences.
Decimal-Octal Conversion table 1000 values and differences.
Tables of binary sines and cosines to 32b. (For machine use).
Chebyshev polynomial co-efficient evaluation.
Solution of sets of simultaneous equations (up to 16 × 16).
Determination of a crystal structure.

Perhaps the most interesting of these applications is the last, the structure under investigation was that of Oxalic Acid Dihydrate using full three dimensional data. The whole analysis required less than 2 weeks, during which time the machine evaluated the equivalent of six sets of three dimensional Fourier syntheses (each at 48 points) together with three evaluations of the expression:[1]

$$F_c(hkl) = 2\sum_{v=1}^{8} f_v \cos 2\pi\left(\frac{hx_v}{a} + \frac{ky_v}{b} + \frac{lz_v}{c}\right)$$

each for 1200 values of (hkl). It is estimated that this work would have taken a research student between 6 and 12 months by ordinary methods.

After its move from London to Manchester the machine was in operation within one week and it has lately been in use in statistical problems concerned with mill productivity.

Whilst A.P.E.(R).C. was being tested several other machines of the same general design were constructed. The first group of these were based upon a redundancy analysis of A.P.E.(R).C. which resulted in the elimination of 180 vacuum tubes from the original system, they are:

A.P.E.(X).C.	at Birkbeck College Computation Laboratory.
A.P.E.(N).C.	at the Norwegian Board for Computing Machines, Oslo.
M.A.G.I.C.	(Magnetic and Germanium Integer Calculator) at Wharf Engineering Laboratories, England.

In the first and last of these machines the storage capacity has been increased to over 8000 words of 32 bits, and the repetition frequency doubled (60 K.C./sec.). Input/output are either by teletype or by punched card.

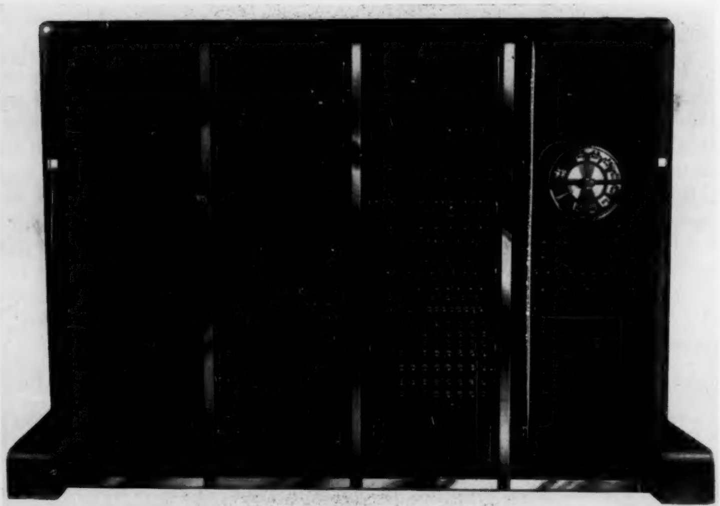

FIG. 2. Hollerith Electronic Computer. 2.

The second group of machines are based upon the original A.P.E.(R).C. design and have been constructed by the British Tabulating Machine Company Ltd., they are noticeable for the provision of a special input conversion device,[9] which enables them to work efficiently in conjunction with decimally punched cards. H.E.C.1. (Hollerith Electronic Computer 1) has been in use for program investigation for over 12 months and H.E.C.2. was exhibited at the Business Efficiency Exhibition, London, in June 1953, giving fault-free operation for $99\frac{3}{4}\%$ of the four days of the show.

A photograph of H.E.C.2., which is now available commercially, is shown in Fig. 2.

3. Code and Control Word Layout on A.P.E.(X).C.—Although minor differences both in the actual control word constitution, and in the orders themselves, exist between the various members of the A.P.E.(X).C. and H.E.C. families; the scope of these machines will be evident from the following description which refers to A.P.E.(X).C. and M.A.G.I.C.

Digits	0 to 5	6 to 10	11 to 15	16 to 20	21 to 24	25	26 to 31	32
	X Track	*X* Location	*Y* Track	*Y* Location	Code	Spare	C6	Spare

Make up of Instruction Word in A.P.E.(X).C.

A.P.E.(X).C. Code

No.	Symbol	Description
0	$(y)E$	Stop.
1	$(y)H$	Card feed.
2	$(y)P$	Print first 5 binary digits of number held in register.
3	$(x)(y)Cc$	If number in $A \geqslant 0$ select next order from position (y) if $A < 0$ select instruction from (x).
4	$(y)l(n)$	Left shift n places. A and R are connected cyclically.
5	$(y)r(n)$	Right shift n places. $A(0).A(1)\cdots A(31)$; $R(0), R(1)\cdots R(31)$ becomes $A(0).A(0)A(1)\cdots A(30)$; $A(31), R(0), R(1)\cdots R(30)$, etc.
6		Spare.
7		Spare.
8	$(x)(y)+c$	Add contents of (x) into cleared accumulator.
9	$(x)(y)-c$	Subtract contents of (x) from cleared accumulator.
10	$(x)(y)+$	Add contents of (x) into accumulator.
11	$(x)(y)-$	Subtract contents of (x) from accumulator.
12	$(x)(y)+R$	Transfer contents of (x) to R.
13	$(x)(y)\times(n)$	Multiply number in (x) by last n digits of number in R. First half of product in A last part in R.
14	$(x)(y)\ A(n)$	Record first (or last) n digits of A in (x).
15	$(x)(y)\ R(n)$	Record first (or last) n digits of R in (x).

Some features of this code require further description. In the first place it will be noticed that each order has both an (x) and a (y) address. The first (x) refers to the location of the number required in the performance of the operation, whilst the second (y) gives the location of the next instruction. The number (n) which specifies shift, precision of multiplication, and numbers of digits to be recorded has to be placed in digits 26–31 (Marked C6).

It is chosen in accord with the following rules:

$l(n)$	C6 contains (n)
$r(n)$	C6 contains $(64\text{-}n)$
$\times(n)$	C6 contains $(64\text{-}n)$
$A(n)$	C6 contains $(32\text{-}n)$ if 1st (least significant) n digits of A are to be recorded, but $(64\text{-}n)$ if first n digits are to be recorded.
$R(n)$	C6 as in $A(n)$

The multiplier is based upon the non-restoring method described elsewhere,[10] and operates upon numbers of any sign; it normally short-cuts and the provision of the (n) digit facility is an aid to programming rather than to speed.

The times of the various arithmetic operations (in A.P.E.(X).C.) are:

$+, -,$	600μ sec.
$\times$	$m \times 600\mu$ sec.
$r(n)$	600μ sec.
$l(n)$	$1{,}200\mu$ sec.

In these figures no account has been taken of access; with efficient optimum programming it is found that an increase by a factor of only slightly greater than two is appropriate, but it must be remembered that by "non-optimum" programming all of these figures can be increased by a factor of up to 32! It will be seen that the multiplication time is given as $m \times 600\mu$ sec. The factor m represents the number of changes 0 to 1 or 1 to 0 of the digits in the multiplier.[10]

A point worthy of mention is that the correct working of the machine in no way depends upon the accuracy of the optimum programming, if the coder makes a mistake in this it will not cause a machine fault but merely a waste of time. This is a feature which is absent in at least one other machine employing a multi-address code. To complete the description, it must be mentioned that each track on the drum contains 32 words, and it is within these that optimisation is required since inter-track switching is performed between words and causes no delay.

The storage (8×32 tracks each of 32 words) is at present available only in blocks of 32 tracks and the programmer has to select and plug those tracks which he will require in a given problem. This limitation may or may not be temporary, it is in contemplation to have a telephone director type system which will perform the track selection upon instruction either from the machine, or from punched cards; the building of this is, however, awaiting the result of coding studies which are still in progress.

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[1] A. D. BOOTH, *Fourier Technique in X-ray Organic Structure Analysis*. Cambridge, 1948.
[2] A. D. BOOTH & K. H. V. BRITTEN, *Coding* for A.R.C. Mimeographed. Princeton, 1947.
[3] A. D. BOOTH, "A magnetic digital storage system," *Electronic Eng.*, v. 21, 1949, p. 234–238.

[4] N. Kitz, *Thesis for the degree of M.Sc.* University of London, 1951.
[5] A. D. Booth & K. H. V. Booth, *Automatic Digital Calculators*, London, 1953.
[6] A. D. Booth, "The physical realization of an electronic digital computer," *Electronic Eng.*, v. 22, 1950, p. 492–498.
[7] A. D. Booth, "The physical realization of electronic digital computer," Part II. *Electronic Eng.*, v. 24, 1952, p. 442–445.
[8] A. D. Booth, "On optimum relations between circuit elements and logical symbols in design of electronic calculators." Brit. I.R.E. Jn., v. 12, 1952, p. 587–594.
[9] R. Bird, "Computing machines—Input and output," *Electronic Eng.*, v. 25, 1953, p. 407–409.
[10] A. D. Booth, "A signed binary multiplication technique," *Q. Jn. Mech. Appl. Math.*, v. 4, 1941, p. 236–240.

Bibliography of Coding Procedure

The material described below is among that which has been added to the collection at the National Bureau of Standards Computation Laboratory. A similar collection is available at the National Bureau of Standards' Institute for Numerical Analysis.

Material for inclusion in these collections should be sent to the Computation Laboratory, National Bureau of Standards, Washington 25, D. C., for the attention of J. H. Wegstein.

35. University of Toronto, *FERUT*, *Program Notes, Library Routines.*

This is an expandable library of FERUT routines preceded by chapters from the *Programmers' Handbook for the Manchester Electronic Computer* and notes on the FERUT library.

36. Ferranti Ltd., *Introduction to programming, notes, and routine specifications for the Manchester Electronic Digital Computer.*

This booklet was prepared as an introduction to programming to facilitate the study of the *Programmers' Handbook for the Manchester Computer Mark II*. It is prepared in loose-leaf book form so that *Ferranti Routine Specifications* can be added as they are issued.

37. The Computing Machine Laboratory, The University of Manchester, *Programmers' Handbook (2nd Edition) for the Manchester Electronic Computer Mark II.*

This contains chapters on the logical design of the machine and the instruction code, coding examples, the preparation of a problem with many programming details, aids to coding (e.g., translation routines, interpretive routines), the calculation of functions of a single variable, numerical solution of ordinary differential equations, fault diagnosis in programmes, measures against machine breakdown, and the console.

38. Klaus Jamelson & Frederich L. Bauer, "Optimale Rechengenauigkeit bei Rechenanlagen mit gleitendem Komma," *Z. angew. Math. Phys.* v. 4, 1953, p. 312–316.

In most machines with floating decimal (or binary) point a complete "zero-shift" or "normalization" is carried out either immediately after each operation (Mark II, Bell-computer) or before a multiplication (BARK), i.e., the numbers are shifted to the left until the first non-vanishing digit occupies the leftmost position. This method has the drawback that occasionally (e.g., after a subtraction of two nearly equal numbers) insignificant figures, due to round-off etc., are shifted far to the left, where they may spoil subsequent operations. The paper under review proposes to make incomplete shifts in such cases, keeping the

insignificant figures at the right. The allowable amount of shift and how it is determined by the machine itself is discussed. For binary computers, since the first digit of a mantissa is now not necessarily different from zero, a modification of the usual conversion routines becomes necessary, and this is discussed in detail.

PETER HENRICI

NBSCL

39. T. PEARCY (Commonwealth Scientific and Industrial Research Organization (CSIRO)), *Automatic Computation Part I, The Design of the Mk. I Automatic Computer.*

This is a description including logical block diagrams of the design of the Mk. I computer. It is a 1024 word, mercury delay-line memory machine.

40. T. PEARCEY (C.S.I.R.O.), *Automatic Computation Part II, Programmes for an Automatic Computer.*

This is an introductory programmer's manual for the CSIRO Mk. I computer.

41. T. PEARCEY (C.S.I.R.O.), *Automatic Computation Part III, Programmes for the Mk. I Computer: Pt. I.*

This is an extension of Part II and discusses the preparation of subroutines for the Mk. I computer.

42. IBM SCIENTIFIC COMPUTING SERVICE, *IBM 701 Speedcoding System.*

The "speedcoding system" refers to the IBM 701 equipment and a method in which the computer aids in the preparation of its own programs. The system permits the programmer to supply the binary machine with input data in decimal form and with a pseudo-code. The pseudo-code is more compact and can be more easily and accurately written than the true machine code. The objective of this method is to shorten the time required in preparing problems for solution with the high speed computer.

Speedcoding utilizes a floating point system. The pseudo-code includes the regular instructions of the true code but also includes single instructions which designate computation of square root, evaluation of trigonometric and exponential functions, certain tape and drum manipulations, and error checking operations.

This booklet is a manual for users of the speedcoding system.

J. H. WEGSTEIN

NBSCL

43. R. R. HARE, *Coding for the Florida Automatic Computer (FLAC)* (revised April 1953).

This report is intended as a reference for the coding of problems for the Florida Automatic Computer at Patrick Air Force Base, Florida. It briefly describes the floating three-address system employed by the FLAC, together with the instructions which may be performed by this computer.

44. MATHEMATICS PANEL, OAK RIDGE NATIONAL LABORATORY, *Manual for the ORACLE.*

This beautifully printed and handsomely bound manual is expandable. It presently contains the following sections: "Electronic Digital Computers" by C. L. PERRY, a general description of digital computers; "Basic Operations of the ORACLE" by A. S. HOUSEHOLDER, which includes a list of the ORACLE operations; "Flow Charts and Coding" by J. MOSHMAN, which explains flow diagramming and gives examples of flow diagrams and their corresponding codes.

45. H. RUTISHAUSER, *Automatische Rechenplanfertigung bei programmgesteuerten Rechenmaschinen.* Mitteilungen aus dem Institut für angewandte Mathematik an der Eidgenössischen Technischen Hochschule in Zürich, No. 3. 45 p., Basel 1952.

A considerable part of the time consumed by the solution of a problem on an automatic computing machine is usually spent for coding. Various attempts have therefore been made to mechanise not only the computation, but also the coding procedure (Coding machine for Mark III, K. ZUSE's "Plankalkül" etc.). All these methods require special equipment, and the computing machine is not used for the coding. Contrary to this, in the report under review H. RUTISHAUSER gives a thorough and complete description of his method to "compute" the code for a given problem on the computing machine itself. Although this method is in principle quite general and could, with some modifications, be carried out on any automatic computing machine, the present exposition is based upon the special properties of the machine to be built at the Institute for Applied Mathematics in Zürich.

The first section of the report is devoted to a short description of this machine, the plans of which have grown out of the experience of its planners with Zuse's and Aiken's machines. It is a one-address machine with an internal drum memory of 1000 cells. Instructions are stored in a separate memory of the same capacity, each cell holding two instructions. Orders can be modified by means of nine index registers. A special order is provided for the punching of an order computed on the machine.

The second section is the heart of the paper. It describes the computation of a code for the evaluation of a given algebraic expression ("bracket expression"). Associated with every symbol E_k of a bracket expression (brackets and operation symbols included) are two integers, a_k and b_k. Roughly speaking, the more in the interior a symbol is located in the bracket expression, the larger is the corresponding number a_k. By an ingenious algorithm the bracket expression is then decomposed step by step, starting from the interior-most elements, and the corresponding orders are computed.

The third section deals with cyclic problems. Here one and the same bracket expression is to be evaluated several times and for several values (and/or addresses) of the numerical parameters. The author proposes to "stretch" the code, i.e., to unfold the cycles and to compute and to store the instructions for each iteration step separately. It is true that some computation time is thus saved by avoiding a decision as to the correct alternative after completion of each cycle. In view of the obvious limitations of memory space, the procedure nevertheless does not seem to be quite convincing.

In the fourth and last section an example is given for the computation of a code for a cyclic problem without unfolding. Here the general method is complicated by a considerable amount of special rules and provisions. Only experience can show if the author's method results in a real gain over straight-forward coding by hand.

In the whole this is a remarkable publication, which should provide a sound basis for further researches in the theory of automatic coding.

PETER HENRICI

NBSCL

46. National Bureau of Standards *Report*, September 30, 1952. *SEAC Operating and Programming Notes*, V.

26. Interpretive Subroutine for Operations on Complex Numbers
27. Subroutine for Square Root of a Single Precision Number with Floating Binary Point
28. Subroutine for Sinh y, Cosh y, $|y| \leq 2.0634$
29. Code Checking
30. Basic Arithmetic Operations for Double Precision Numbers with Fixed Binary Point
31. Interpretive Subroutine for Operations on Double Precision Numbers

47. National Bureau of Standards *Report*, April 13, 1953. *SEAC Operating and Programming Notes*, VI.

32. Interpretive Subroutines S36 and S37 for Operations on Single Precision Numbers in Floating Binary Point Form, and Breakpoint Subroutine S38
33. Subroutine for Double Precision, Floating Decimal Point Operations
34. Subroutine for $\text{Log}_e N$; Single Precision, Floating Binary Point
35. Subroutine Assembly Routine (SASS)
36. Tape Control Subroutines S21 and S22

48. National Bureau of Standards *Report*, April 24, 1953. *SEAC Operating and Programming Notes*, VII.

37. Basic Arithmetic Operations II, Floating Binary Point, Single Precision Numbers
38. Interpretive Subroutine for Operations on Double Precision Numbers with Floating Decimal Point
39. Subroutine for the Square Root of a Double Precision Number with Fixed Binary Point
40. Subroutine for the Square Root of a Single Precision Number with Floating Binary Point II
41. Automonitor Processing Routine (BIG BOIE)

49. National Bureau of Standards *Report*, November 13, 1953. *SEAC Operating and Programming Notes*, VIII.

42. Subroutines S30, S31, S32, and S33 for the step-by-step integration of the first order differential equation $y' = f(x, y)$
43. Subroutine for 16-point Gaussian quadrature
44. Subroutine for binary to decimal conversion of a double precision number with fixed binary point

45. Subroutine for binary to decimal conversion of a single precision number with floating binary point
46. Subroutine for decimal to binary conversion of a single precision number with floating binary point
47. Subroutine for conversion of decimal degrees to binary radians and vice versa
48. Subroutine for the evaluation of polynomials, single precision, fixed binary point
49. Automatic transfer routine

50. NBS INSTITUTE FOR NUMERICAL ANALYSIS. *SWAC Memoranda.*

SWAC Memoranda are prepared in order to maintain a record of current developments in SWAC operating and programming. These memoranda include routines of general interest and applicability, facts of interest to programmers concerning modifications or newly installed facilities of the SWAC, and other information of general importance. The information incorporated in these memoranda represent the efforts being made in the Institute to develop techniques (a) for reducing the amount of time that goes into code preparation and checking, (b) for checking machine and auxiliary equipment operations, (c) for mechanizing as much as possible the code preparation process, and (d) for making optimum use of SWAC equipment.

Memoranda in this series will in general be reviewed by title only.

2. Interpretation Routine
3. General Test Routine
4. The Magnetic Drum
5. Break Point Digit
6. Relative (Symbolic) Coding
7. Diagnostic Extract Routine
8. Memory Test Series 00000 to 00099

51. L. J. PAIGE & C. B. TOMPKINS, *Systematic generation of permutations on an automatic computer and an application to a problem concerning finite groups.* NBS working paper, January 30, 1953.

The authors give a method for generating permutations on n marks which they have coded for SWAC for $n = 7,9$. The method is as follows: Suppose that all permutations on $n - 1$ marks are known. If $(i_1, i_2, \cdots, i_{n-1})$ is such a permutation, then $(i_1, i_2, \cdots, i_{n-1}, n - 1)$ and all its cyclic permutations will generate all permutations on n marks. The method is applied to the problem of determining all "complete" mappings of a certain group of order 8 onto itself. An alternative method of generating permutations due to MARSHALL HALL is discussed. A detailed code is included.

MORRIS NEWMAN

NBSCL

52. ACM, *Proceedings of the meeting at Pittsburgh, Pa.*, May 2–3, 1952.

Of the 41 papers in these proceedings the following are noted under this bibliography:

"Construction and Use of Subroutines for the SEAC," J. H. LEVIN.

"The Use of Subroutines on SWAC," ROSELYN LIPKIS.

"The Use of Subroutines in Programmes," DAVID J. WHEELER.
"Progress of the Whirlwind Computer Towards an Automatic Programming Procedure," JOHN W. CARR.
"The Education of a Computer," GRACE M. HOPPER.

53. ACM, *Proceedings of the meeting at Toronto, Ontario*, September 8–10, 1952.

Of the 35 papers in these proceedings the following are noted under this bibliography:
"Compiling Routines," R. K. RIDGWAY.
"Machine Aids to Coding," E. J. ISAAC.
"Computer Aids to Code Checking," I. C. DIEHM.
"Input Scaling and Output Scaling for a Binary Calculator," E. F. CODD, H. L. HERRICK.
"Logical or Non-Mathematical Programmes," C. S. STRACHEY.
"A Simplified Universal Turing Machine," E. F. MOORE.
"Simple Learning by a Digital Computer," The COMPUTATION LABORATORY, Harvard University.
"Use of Continued Fractions in High-Speed Computing," D. TEICHROEW.
"Interpretive Sub-Routines," J. M. BENNETT, D. G. PRINZ, M. L. WOODS.
"The Use of Sub-Routines on SEAC for Numerical Integrations of Differential Equations and for Gaussian Quadrature," P. RABINOWITZ.
"Pure and Applied Programming," M. V. WILKES.

54. NATIONAL PHYSICAL LABORATORY, Teddington, England. *Automatic Digital Computation Symposium*, March 25–28, 1953.

This is a collection of copies of most of the 37 papers presented. The following are noted under this bibliography:
"Optimum Coding," G. G. ALWAY.
"Micro Programming and the Choice of Order Code," J. G. STRINGER.
"Conversion Routines," E. N. MUTCH.
"Getting Programmes Right," S. GILL.
"Diagnostic Programmes," R. L. GRIMSDALE.

55. OSBORNE, E. E., *Solution of the Matrix Equation* $(M - \Omega D)X = 0$. NBS report, October 16, 1953.

The author gives a solution by the power method (successive matrix-vector multiplication and vector normalization) of the equation of the title, M being a 6×6 matrix having complex elements and D a non-singular real diagonal matrix. Setting $A = D^{-1}M$, the equation becomes the more familiar one

$$(A - \Omega I)X = 0,$$

and this is the one the author actually solves.

The preliminary multiplication $D^{-1}M$ was performed on IBM equipment. The computation of the eigenvalues Ω and eigenvectors X was programmed for SWAC. Detailed flow charts and codes are given for two such programs, one which can be stored in the high speed memory, the other requiring external storage capacity such as a magnetic drum.

MORRIS NEWMAN

NBSCL

BIBLIOGRAPHY Z

1114. ANON., "New knowledge—computers are busy these days," *Westinghouse Engineer*, Jan. 1954, p. 62–63.

Industry is finding computers economical as well as efficient. An Eastern utility is using one to establish power-system loss formulas and data for load dispatching on large interconnected systems at a saving of $60,000 yearly.

They are being used for the automatic solution of problems involved in the routine application of electrical machinery; problems in connection with blades of compressors and steam turbines; reduction of data on all effects of inner-cooled generators and stress-analysis problems on rigid steam piping.

A. R. COCK

NBSCL

1115. ANON., "New digital computer speeds math solution at Oak Ridge," *Industrial Laboratories*, Nov. 1953, p. 128–129.

The Oak Ridge Automatic Computer Logical Engine, ORACLE, is described in 9 paragraphs and 3 pictures. It is a descendent of the Institute for Advanced Studies Computer, is fully parallel, and has cathode-ray tubes storage for 2048 words. Net addition time is 5 microseconds and multiplication time 500 microseconds.

R. D. ELBOURN

NBSCML

1116. ANON., "Utilization of digital computing machines," *Nature*, v. 172, 1953, p. 649–651.

This is a report of a meeting of the Engineering Section of the British Association in September 1953. The speakers were F. C. WILLIAMS, R. K. LIVESLEY and G. G. ALWAY. It is interesting to note British experience and opinions in this field: for instance, that at present the need is for more programmers and programs rather than for more machines, that it takes about three months to train a programmer and about the same time to prepare a significant problem, and that the speed factor is 150:1 over hand computing. The desirability of making the machine carry out as much as possible of the preparation of data, no matter how trivial it is, is stressed. It is also noted that the knowledge of the capabilities of computers gained during the solution of one problem has often led to customers realization of the feasibility of attacking other problems in their field.

Among the problems which have been handled (on the machines at University of Manchester and at the National Physical Laboratory) are: torsional and lateral oscillations of systems of shafts, design of cams, analysis of rigid-frame structures, flutter analysis.

J. T.

1117. P. R. BAGLEY, *Electronic Digital Machines for High-speed Information Searching*. Master's Dissertation, Mass. Inst. of Tech., Cambridge, 1951, ix + 133 p.

The author's objective is the examination of the suitability of methods and machines for the high-speed location of related items in a large body of

indexed information. His study is a timely one because of the mountainous volume of information, growing at an ever-increasing rate, which exists in the scientific and engineering fields. For example, a search of *Chemical Abstracts* today involves over a million abstracts, with the present trend of publication indicating that the total abstracts published in this field by 1960 will be almost 1,800,000. In another representative field, Electrical Engineering, it has been estimated that by 1960 the total of published papers of interest may reach 700,000.

Since the effective planning of research requires a knowledge of previous activity in the field, it is apparent that the increasing size of our "storehouse" of information makes it imperative that we attempt to increase the speed of our information searching facility. A notable stride in this direction has been taken in the development of the Rapid Selector, designed by DR. VANNEVAR BUSH, which is in use and being developed further, by the U. S. Department of Agriculture Library. In respect to the problem of information searching, methods and techniques for searching inventory records are interesting. For example, in the October 1953, *Proceedings* of the I.R.E., there was reported an investigation of photographic techniques for information storage underway by the International Telemeter Corporation, which could result in further alleviation of the information searching problem.

The thesis is divided into three parts: Part One, Introduction to the Information Searching Problem; Part Two, Investigation of Electronic Digital Methods for High-Speed Scanning and Selection; Part Three, Conclusions and Recommendations. Four Appendixes are added: Appendix 1, Derivation of Superposed Coding Formulas; Appendix 2, Description of Major Electronic Computer Components; Appendix 3, Guide to Coding for the Whirlwind I Computer; Appendix 4, Searching with the Whirlwind I Computer. A thorough analysis of the problem prefaces an attempt at solving it by use of a high-speed electronic digital computer.

The conclusion of the author is that the general-purpose digital computer is virtually disqualified as a selection device because of its sequential operation and the fact that, in the application, substantial time is required for the transfer of numbers back and forth between the storage element and the arithmetic element of the machine. He considers it feasible to develop an electronic computer more suitable for information searching than the general-purpose digital computer, and presents a block diagram for such a device.

E. W. C.

1118. H. BÜCKNER, F. J. WEYL, L. BIERMANN & K. ZUSE, *Probleme der Entwicklung programmgesteuerter Rechengeräte und Integrieranlagen*. Rhein.-Westf. Technische Hochschule Aachen, Mathematisches Institut, H. CREMER, ed., Aachen, 1953, xiii + 75 p.

This book contains the essential contents of four speeches given at a colloquium in Aachen in July 1952, under the sponsorship of the Institute for Mathematics, Mechanics, Physics and Theoretical Physics of the Rheinish-Westfälischen Technischen Hochschule.

The four chapters of the book, based upon the speeches of the authors in the order above, are the following: Über die Entwicklung des Integromat; Aufbauprinzip, Arbeitsweise und Leistungsfähigkeit elektronischer pro-

grammgesteuerter Rechenautomaten und ihre Bedeutung für die naturwissenschaftliche Forschung; Die Gottinger Entwicklungen elektronischer Rechenautomaten; Über programmgesteuerte Rechengeräte für industrielle Verwendung.

In the first chapter, Bückner gives a functional description of the Integromat, a differential analyzer. In the second chapter, by F. J. Weyl, the characteristics of electronic digital computers now in use in the United States are compared and some of their uses are discussed. Included in the comparison of machines is the primary motivation of their construction—ballistics computations, the development of improved components and more flexible computational systems, or commercial and industrial applications. In the third chapter, L. Biermann discusses the features of a new digital computer in operation at the Max-Planck Institute of Physics. The computer is a moderate-speed, tape-fed, magnetic drum machine, operating on 32 binary-digit numbers, with a 3 binary-digit integral part. Some of the problems being solved on the computer are described. The last chapter, by Zuse, is an exposition of his development of relay computers, prefaced by a review of the difficulties he encountered up to the time of the installation of a Zuse computer in the Eidgenössische Technische Hochschule at Zürich in 1950. He presents an interesting explanation of the reasoning underlying many of his design decisions, and states a case for moderate speed, simple-to-operate computers, particularly for application to engineering problems.

The book closes with a recording of the discussion which ensued after the presentation of the papers.

E. W. C.

1119. Office of Naval Research, *A Survey of Automatic Digital Computers*. Washington, 1953, vi + 109 p. Available as PB111293, U. S. Dept. of Commerce. Office of Technical Services, Washington 25, D. C. Price $2.00.

This compendium of data on automatic digital computers incorporates the results of surveys by the Flight Research Laboratory, Wright-Patterson Air Force Base, and the Mathematical Sciences Division, Office of Naval Research.

Included in the survey is one page of information on each of ninety-eight computers, among which are included machines in use, or under development, in Australia, Belgium, Canada, England, France, Germany, Holland, Japan Norway, Sweden, Switzerland and the United States.

A partial listing of the information given on the computers included in the survey is the builder, location of installations, availability of programming service and computing time, operating schedule, number base, word length, instruction type, sequence control, built-in operations and time required for their execution, description of internal storage and listing of input-output devices.

The Office of Naval Research survey is a useful reference for workers in the automatic digital computer field and also for those who, though not concerned with their development, are interested in the application of high-speed digital computers to their problems.

E. W. C.

1120. Mina Rees, "Applying Computers to Machine Control," *Machine Design*, v. 25, 1953, p. 324–334.

An abstract of an address entitled "The Future Use of Digital Computers" given at Eighteenth National Applied Mechanics Conference of the American Society of Mechanical Engineers, Minneapolis, June 1953.

News

Eastern Joint Computer Conference and Exhibition.—The Joint Computer Conference on Information Processing Systems—Reliability and Requirements was held at the Statler Hotel in Washington, D. C., December 8–10, 1953, under the joint sponsorship of the American Institute of Electrical Engineers, the Institute of Radio Engineers, and the Association for Computing Machinery.

Session I Tuesday, December 8, 10:15 AM

Chairman—F. J. Maginniss, Secretary, AIEE Committee on Computing Devices

Address of Welcome	John H. Howard, *Chairman*, IRE Professional Group, Electronic Computers
Keynote Address	Howard T. Engstrom, Remington Rand, Inc.
The RTMA Support of the 1950 Computer Conference—A Progress Report	Thomas H. Briggs, Burroughs Corporation
The Use of Electronic Data Processing Systems in the Life Insurance Business	M. E. Davis, Metropolitan Life Insurance Company
Computer Applications in Air Traffic Control	Vernon I. Weihe, Air Transport Association of America
Discussion of papers	

Session II Tuesday, December 8, 2:00 PM

Chairman—Howard T. Engstrom, Remington Rand, Inc.

Data Processing Requirements for the Purposes of Numerical Weather Prediction	Joseph Smagorinsky, U. S. Weather Bureau
Methods Used to Improve Reliability in Military Electronics Equipment	L. D. Whitelock, Bureau of Ships
Digital Computers for Linear, Real-Time Control Systems	Ralph B. Conn, Jet Prop. Lab., California Institute of Technology
The MIT Magnetic-Core Memory	William T. Papian, Massachusetts Institute of Technology
Discussion of papers	

Session III Wednesday, December 9, 9:00 AM

Chairman—C. V. L. Smith, Office of Naval Research

Reliability Experience on the OARAC	Robert W. House, Wright-Patterson Air Force Base
Operating Experience with the Los Alamos 701	William G. Bouricius, Los Alamos Scientific Laboratory
Acceptance Tests for the Raytheon Hurricane Computer	Francis J. Murray, Columbia University
Reliability of a Large REAC(R) Installation	Bernard D. Loveman, Reeves Instrument Corporation
National Bureau of Standards Performance Tests	S. N. Alexander & R. D. Elbourn, National Bureau of Standards
Experience on the Air Force UNIVAC	Robert Kopp, Headquaters, U. S. Air Force
Discussion of papers	

Session IV Wednesday, December 9, 2:00 PM

Chairman—C. C. Bramble, U. S. Naval Proving Ground, Dahlgren

Electron Tube and Crystal Diode Experience in Computing Equipment	J. A. GOETZ & H. J. GEISLER, IBM Corporation
Reliability and Characteristics of the ILLIAC Electrostatic Memory	JOSEPH M. WIER, University of Illinois
Electron Tube Performance in Some Typical Military Environments	D. W. SHARP, Aeronautical Radio, Inc.
Discussion of papers	
Session V Thursday, December 10, 9:00 AM	
Chairman—R. F. CLIPPINGER, Raytheon Manufacturing Company	
SEAC—Review of Three Years of Operation	P. D. SHUPE, JR., & R. A. KIRSCH, National Bureau of Standards
A Review of ORDVAC Operating Experience	CHARLES R. WILLIAMS, Ballistic Research Laboratory
Some Remarks on Logical Design and Programming Checks	HERMAN H. GOLDSTINE, The Institute for Advanced Study
The Advantages of Built-in Checking	JOHN W. MAUCHLY, Remington Rand, Inc.
Recent Progress in the Production of Error Free Magnetic Computer Tape	J. C. CHAPMAN & W. W. WETZEL, Minnesota Mining and Manufacturing Company
Discussion of papers	
Session VI Thursday, December 10, 2:00 PM	
Chairman—S. B. Williams, President, ACM	
Reliability of Electrolytic Capacitors in Computers	MARK VAN BUSKIRK, P. R. MALLORY and Company, Inc.
Reliability and its Relation to Suitability and Predictability	E. B. FERRELL, Bell Telephone Laboratories
Case Histories in Resistor Stability	JESSE MARSTEN, International Resistance Company
Discussion of papers	
Summary	ALLEN V. ASTIN, Director, National Bureau of Standards

Group discussions were held each day at 4 PM which included the following topics:

Wednesday, December 9	
Applications—Technical	GEORGE PETRIE, *Chairman*, International Business Machines Corp.
Applications—Commercial and Industrial	WALTER FRESE, *Chairman*, General Accounting Office
Diagnostic Checks	J. J. EACHUS, *Chairman*, Dept. of Defense
Magnetic Tape Standards	ALLEN SHOUP, *Chairman*, Shoup Engineering Co.
Crystal Diodes (design for reliability)	RALPH J. SLUTZ, *Chairman*, National Bureau of Standards

OTHER AIDS TO COMPUTATION

1121. W. G. ANDERSON & E. H. FRITZE, "Instrument approach system steering computer," I. R. E., *Proc.*, v. 41, 1953, p. 219–228.

This paper is mainly devoted to a theoretical discussion of a computer to be used in providing steering indications to an aircraft pilot during a blind landing using the Instrument Landing Approach System. The purpose of the computer considered is to combine signals from three sources used by a pilot in flying such an approach. These sources are the vertical gyro (bank

angle), the directional gyro (deviation of heading from runway direction) and localizer receiver (lateral deviation from desired flight path).

The approach procedure is interpreted as a feedback control process. The characteristics of the data sources and the aerodynamic characteristics of the aircraft are considered in order to devise a steering indication capable of yielding suitable approaches. In particular, a system for deriving a desirable signal for rate of change of lateral deviation is described.

The computer is described briefly. It is of conventional 400-cycle analog type. The use of vacuum tubes is held to a minimum and limiters are employed to prevent indication of excessive bank angles.

L. H. O'NEILL

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New York, New York

1122. A. R. BOOTHROYD, "Design of electric wave filters with the aid of the electrolytic tank," Inst. Elect. Eng., *Proc.*, v. 98, part IV, 1951, p. 65–93.

This article is intended as a "treatise" on the subject. An appendix gives details of tank construction.

1123. JOHN BROOMALL & LEON RIEBMAN, "A sampling analogue computer," I. R. E., *Proc.*, v. 40, 1952, p. 568–572.

The analogue computer described in this article is based on an idea of C. J. HERSCH and J. F. FELKER. The input of the computer consists of three steady d. c. voltages with values X, Y and Z. The computation is based on the sampling of d. c. voltages by means of electronic switching circuits using diodes. The computer has two units; the first of these is an algebraic unit, which produces a voltage pulse whose amplitude W is given by $W = kZYX^{-1}$. The second unit produces a steady voltage U in the form $aZYX^{-1} + b$ for constants a and b. In the algebraic unit the voltages Z and X are sampled and the condensers of identical RC networks are charged to these voltages. At a time t after this sampling, the two condensers have voltages $X\exp(-t/RC)$ and $Z\exp(-t/RC)$. The first of these voltages is compared by an electronic circuit with the voltage Y and when these are equal a sample pulse whose size is proportional to the second voltage is emitted by this unit. At the time t_0 when the first of these voltages equals Y, $\exp(-t_0/RC) = Y/X$, and hence at this time the second voltage has the value ZYX^{-1} up to a constant. The voltages involved have a full-scale range of 0 to 150 volts and accuracy of 1% of full scale is claimed. The "settling time" of the pulse converter is given as 30 milliseconds and presumably this is the limit on the repetition rate of the unit.

F. J. M.

1124. E. H. FRITZE, "Punched card controlled aircraft navigation computer," I. R. E., *Proc.*, v. 41, 1953, p. 734–742.

The computer described is for the purpose of enabling an aircraft pilot to fly to an arbitrarily selected point. It provides indications of the heading to be flown, the distance to the selected point and the displacement of the selected point from a "master" ground station. In addition, the equipment

indicates the directions from the aircraft to the master station and an auxiliary station. In flight test, the distance indications obtained during a 200 mile flight were in error by less than 2½ miles.

Input data to the computer are obtained from radio receivers which indicate the directions to a "master" and an auxiliary omnirange station. Such stations have been installed throughout the United States by the Civil Aeronautics Administration (CAA). The device is also capable of employing as input data the range and bearing of a single dme omnirange station. dme is a system planned for future installation by the CAA. Through its use, an aircraft pilot can determine the distance to a station in addition to the direction found by means of the omnirange system.

A substantial amount of auxiliary data is needed by the computer to solve its problem. The problem is basically one of triangulation. A unique feature is that such auxiliary data is stored on plastic punched cards. By positioning the card to reveal the identities of the two stations on which the triangulation is based, a set of switches is caused to insert the auxiliary data. This includes information which tunes the receivers to the selected stations, the rectangular coordinates of one station relative to the other, and a correction to account for the difference in magnetic north at the two stations.

The computer is of conventional analog type employing servomechanisms and resolvers. The basic analogy is that of alternating voltage amplitude to distance. Sign is indicated by the time phase of the alternating voltage with respect to a reference. The most severe limitation on accuracy is the uncertainty in the bearing data provided by the omnirange system.

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1125. R. M. Howe & V. S. Haneman, Jr., "The solution of partial differential equations by difference methods using the electronic differential analyzer," I. R. E., *Proc.*, v. 41, 1953, p. 1497–1508.

There exist a number of methods for solving linear partial differential equations. Frequently one can use separation of variables and reduce the problem to ordinary differential equations, which equations can then be solved on an electronic differential analyzer (cf. references 1–4 of the paper). Another way is to replace the partial differential equation by finite differences in all variables and use an arrangement such as a resistance lattice.[1] In the present article the authors consider various partial differential equations (heat equation in one and two variables, wave equation in one dimension, equation of the transverse vibrations of a beam) and replace the *spatial* derivative by finite differences obtaining a differential-difference equation. For example, in the case of the heat equation

$$C(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K(x)\frac{\partial u}{\partial x}\right] + f(x, t)$$

they obtain

$$C_n(du_n/dt = [K_{n+\frac{1}{2}}/(\Delta x)^2](u_{n+1} - u_n) - [K_{n-\frac{1}{2}}/(\Delta x)^2](u_n - u_{n-1}) + f_n.$$

A circuit is given solving this equation using operational amplifiers and passive elements. Similar differential-difference equations and circuits appear for the other equations considered. Sample solutions are given and comparisons with the solutions obtained by the method of separation of variables are examined.

K. S. MILLER

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[1] F. J. MURRAY, *The Theory of Mathematical Machines*. New York, 1947, p. 77.

1126. R. JENKINS, H. W. BROUGH, & B. H. SAGE, "Prediction of temperature distribution in turbulent flow. Application of the analog computer," *Industrial and Engineering Chemistry*, v. 43, 1951, p. 2483–2486.

A solution obtained on the California Institute of Technology analogue computer is compared with a calculated result. The agreement is described as "fair."

1127. E. F. JOHNSON, "A pneumatic process analog for instruction and research," *Industrial and Engineering Chemistry*, v. 43, 1951, p. 2708–2711.

This instrument is used both for classroom demonstrations and in research as a check against the mathematical analysis.

1128. C. A. MENELEY & C. D. MORRILL, "Application of electronic differential analyzers to engineering problems," I. R. E., *Proc.*, v. 41, 1953, p. 1487–1496.

This is essentially an expository paper. It considers analog electronic differential analyzers of the classical type as well as those involving nonlinear elements. A brief description is given of linear computing elements (operational amplifiers used as summers, integrators, etc.) and their basic computing circuits;[1] nonlinear computing elements such as multipliers and function generators; and input-output elements such as recorders and plotters. Some typical problems in dynamics are outlined mathematically, and closed loop block diagrams for their solution are given. A representative list of applications of electronic differential analyzers is included. Finally, a few brief remarks are made concerning the amount of equipment required for a typical general purpose analyzer.

K. S. MILLER

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New York, N. Y.

[1] J. R. RAGAZZINI, R. H. RANDALL & F. A. RUSSELL, "Analysis of problems in dynamics by electronic circuits," I. R. E., *Proc.*, v. 35, 1947, p. 444–452.

1129. WILLIAM L. MORRIS, "Analogical computing devices in the petroleum industry," *Industrial and Engineering Chemistry*, v. 43, 1951, p. 2478–2483.

Three analog devices are described. One is the Phillips "66 Spectrocomputer," which is a direct current battery device for solving simultaneous linear equations by the Gauss-Seidel method. A second device is a servo de-

vice for solving an algebraic equation. The third is an electrical network analogue for flow problems.

1130. D. M. SWINGLE, "Nomograms for the computation of tropospheric refractive index," I. R. E., *Proc.*, v. 41, 1953, p. 385–391.

1131. P. R. VANCE & D. L. HAAS, "An input-output unit for analog computers," I. R. E., *Proc.*, v. 41, 1953, p. 1483–1486.

The device described can be used as a recorder in plane cartesian coordinates or as a curve tracer type of function generator. The unit consists of a drum on which is wrapped a piece of graph paper. The drum's rotation corresponds to one variable and the axial movement of a carriage carrying either a pen or a potentiometer type transducer corresponds to the other. Movements in both coordinates are controlled by servos and static accuracies within 0.2 per cent of full scale are claimed.

For recording purposes the unit functions in the same way as a conventional plotting board. When used as a function generator the desired function is plotted with conducting ink or a soft lead pencil and attached to the drum. The transducer, which consists of a printed circuit potentiometer card, makes electrical contact with the curve and provides the carriage servo with the error voltage required to follow the curve as the drum turns. The output is furnished by a potentiometer driven by the carriage motion. Errors in following the curve, caused by dynamic limitations of the carriage servo, are compensated by adding the servo error voltage to the output potentiometer voltage. The over-all accuracy seems to be better than one per cent within the range for which the unit is intended. The device was designed for use with the GEDA computer.

ROBERT BERNSTEIN

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1132. G. B. WALKER, "On the electric field in a multi-grid radio valve," Inst. Elect. Eng., *Proc.*, v. 98, part III, 1951, p. 64–67.

The use of electrolytic tanks for this purpose is described.

NOTES

160.—INVERSE INTERPOLATION FOR THE DERIVATIVE IN THE COMPLEX PLANE. In a recent note[1] formulas were given for finding the argument for which a function $f(x)$ has a given derivative $f'(x)$, when that function is tabulated for x at equal intervals h. Those formulas are still applicable when dealing with an analytic function $f(z)$ which is tabulated in the complex plane, so long as the arguments lie equally spaced upon any straight line in the complex plane. But for $f(z)$ tabulated over a Cartesian grid $x + iy$ of length h, greater accuracy may be had by locating the arguments $z_k = z_0 + kh$ closer together by choosing k to be small (generally complex) integers. Thus the problem is to find P, or $z = z_0 + Ph$, when given the values of $f'_z \equiv f'(z) \equiv f'(z_0 + Ph)$ and $f_k \equiv f(z_k)$ at any conveniently located points z_k. We choose the following configurations of points z_k for the

n-point cases, $n = 3(1)7$, where the value of n depends upon the number of points z_k required for direct interpolation:

3—Point	4—Point
$\dot{z}_i$	$\dot{z}_i \quad \dot{z}_{1+i}$
$\dot{z}_0 \quad \dot{z}_1$	$\dot{z}_0 \quad \dot{z}_1$

5—Point	6—Point
	$\dot{z}_{2i}$
$\dot{z}_i \quad \dot{z}_{1+i}$	$\dot{z}_i \quad \dot{z}_{1+i}$
$\dot{z}_0 \quad \dot{z}_1 \quad \dot{z}_2$	$\dot{z}_0 \quad \dot{z}_1 \quad \dot{z}_2$

7—Point

$$\begin{array}{lll} \dot{z}_{2i} & & \\ \dot{z}_i & \dot{z}_{1+i} & \dot{z}_{2+i} \\ \dot{z}_0 & \dot{z}_1 & \dot{z}_2 \end{array}$$

The formula for P in terms of r, s, t, u and v is identical with that for p given previously,[1] namely,

$$\begin{aligned} P = r &- r^2 s + r^3(2s^2 - t) + r^4(-5s^3 + 5st - u) \\ &+ r^5(14s^4 - 21s^2 t + 3t^2 + 6su - v) \\ &+ r^6(-42s^5 + 84s^3 t - 28st^2 - 28s^2 u + 7tu + 7sv) + \cdots. \end{aligned}$$

But the r, s, t, u and v are now defined as follows:

3—Point

$$r = \{2hf'_z + 2(1-i)f_0 - (1-i)f_1 - (1-i)f_i\}/(2D),$$
$$s = t = u = v = 0$$

where $D = -2if_0 + (1+i)f_1 - (1-i)f_i$.

4—Point

$$r = \{2hf'_z + 3(1-i)f_0 + 2if_1 - 2f_i - (1-i)f_{1+i}\}/(2D)$$
$$s = \{3(1+i)f_0 - 3(1-i)f_1 + 3(1-i)f_i - 3(1+i)f_{1+i}\}/(2D),$$
$$t = u = v = 0,$$

where $D = -4if_0 + (3+i)f_1 - (3-i)f_i + 2if_{1+i}$.

5—Point

$$\begin{aligned} r = \{10hf'_z &+ 5(4-3i)f_0 + 20if_1 - (2+i)f_2 - 4(2+i)f_i \\ &- 10f_{1+i}\}/(10D), \end{aligned}$$
$$\begin{aligned} s = \{15(1+3i)f_0 &- 15(5-i)f_1 + 6(1-2i)f_2 + 3(13-i)f_i \\ &+ 15(1-3i)f_{1+i}\}/(10D) \end{aligned}$$
$$\begin{aligned} t = \{-5(1+i)f_0 &+ 10(1-i)f_1 + (1+3i)f_2 - 2(3-i)f_i \\ &+ 10if_{1+i}\}/(5D), \quad u = v = 0 \end{aligned}$$

where $10D = 5(3-11i)f_0 + 20(3+2i)f_1 - 3(3-i)f_2$
$$- 4(9+2i)f_i - 10(3-2i)f_{1+i}.$$

6—Point

$$r = \{20hf'_0 + 40(1 - i)f_0 + 16(1 + 2i)f_1 - (3 - i)f_2 - 16(2 + i)f_i - 20(1 - i)f_{1+i} - (1 - 3i)f_{2i}\}/(20D)$$

$$s = \{45(1 + i)f_0 - 24(2 - 3i)f_1 - 3(2 + 3i)f_2 + 24(3 - 2i)f_i - 54(1 + i)f_{1+i} - 3(3 + 2i)f_{2i}\}/(8D)$$

$$t = \{-4f_0 - 2(1 + 3i)f_1 + f_2 - 2(1 - 3i)f_i + 6f_{1+i} + f_{2i}\}/(D)$$

$$u = \{5(1 - i)f_0 + 4(3 + i)f_1 - (1 - 2i)f_2 - 4(1 + 3i)f_i - 10(1 - i)f_{1+i} - (2 - i)f_{2i}\}/(8D)$$

$$v = 0$$

where $4D = -30if_0 + 32f_1 - (1 - 3i)f_2 - 32f_i + 24if_{1+i} + (1 + 3i)f_{2i}$.

7—Point

$$r = \{20hf'_0 + 4(12 - 11i)f_0 + 40(1 + i)f_1 - (1 - 7i)f_2 - 8(3 + 4i)f_i - 20(3 - i)f_{1+i} + 8if_{2+i} - (3 - i)f_{2i}\}/(20D)$$

$$s = \{9(7 + 9i)f_0 - 12(7 - 15i)f_1 - 33f_2 + 12(12 - i)f_i - 18(3 + 13i)f_{1+i} - 36f_{2+i} - 15if_{2i}\}/(8D)$$

$$t = \{-(17 + 3i)f_0 - 4(4 + 9i)f_1 + (5 - 6i)f_2 - 2(11 - 9i)f_i + 6(7 + 5i)f_{1+i} + 6(1 - i)f_{2+i} + (2 + 3i)f_{2i}\}/(2D)$$

$$u = \{(21 - 13i)f_0 + 20(3 + i)f_1 + (3 + 14i)f_2 + 4(3 - 11i)f_i - 10(9 - i)f_{1+i} + 16if_{2+i} - 3(2 + i)f_{2i}\}/(8D)$$

$$v = \{-3(1 - 3i)f_0 - 12(2 - i)f_1 + 3(2 + i)f_2 + 6(1 + 3i)f_i + 30(1 - i)f_{1+i} - 6(1 + i)f_{2+i} + 3f_{2i}\}/(20D)$$

where $20D = 2(8 - 99i)f_0 + 32(9 - 2i)f_1 + (27 + 31i)f_2 - 96(2 + i)f_i - 40(4 - 7i)f_{1+i} + 32(1 + i)f_{2+i} - (11 - 15i)f_{2i}$.

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[1] H. E. SALZER, "Formulas for finding the argument for which a function has a given derivative," *MTAC*, v. 5, 1951, p. 213–215.

161.—THE NUMERICAL INTEGRATION OF $x''(t) = G(x)$. The formula used by ECKERT, BROUWER, & CLEMENCE[1] in their monumental integration of the equations of motion of the five outer planets, is a modification of one used by COWELL in his investigation of the motion of Halley's comet from 1759 to 1910. In order to get the next position in the ephemeris, use is made not only of those already obtained but of first and second summations.

It seems desirable to have formulas for such work that do not involve summations and such are given here.

Let $x = f(t)$ be a solution of

$$x''(t) = G(x)$$

and put $F(t) = G(f(t))$. We then have

$$f(a + w) = f(a) + wf'(a) + \tfrac{1}{2}w^2f''(a) + \cdots,$$
$$f(a - w) = f(a) - wf'(a) + \tfrac{1}{2}w^2f''(a) + \cdots.$$

We add, express derivatives in terms of *forward* differences, and then replace differences by their values in terms of forward function values, all according to Newton's scheme. Changing w to $-w$, we see that we can write, assuming nth differences of F to be constant,

$$f(a + w) = 2f(a) - f(a - w) + w^2Q,$$

where

$$Q = A_0F(a) + A_1F(a - w) + a_2F(a - 2w) + \cdots + A_nF(a - nw).$$

Let $f(t) = t^2$. Then $F(t) = 2$, and

$$(a + w)^2 = 2a^2 - (a - w)^2 + w^2 2(A_0 + A_1 + A_2 + \cdots + A_n),$$

so that

$$A_0 + A_1 + A_2 + \cdots + A_n = 1.$$

Thus if C is the common denominator of the A's, the sum of their numerators is also C, giving a very simple check.

The following table gives the coefficients for $n = 5(1)9$.

n	5	6	7	8	9
C	240	1 20960	1 20960	36 28800 = 10!	36 28800 = 10!
CA_0	317	1 68398	1 76648	55 37111	57 66235
CA_1	−266	−1 85844	−2 43594	−92 09188	−112 71304
CA_2	374	3 17946	4 91196	213 90668	296 39132
CA_3	−276	−3 11704	−6 00454	−313 23196	−505 69612
CA_4	109	1 84386	4 73136	308 31050	597 00674
CA_5	−18	−60852	−2 34102	−203 32636	−492 02260
CA_6	0	8630	66380	86 46188	278 92604
CA_7	0	0	−8250	−21 48868	−103 97332
CA_8	0	0	0	2 37671	22 99787
CA_9	0	0	0	0	−2 29124
Sum	800	6 79360	12 07360	666 42688	1252 98432
	−560	−5 58400	−10 86400	−630 13888	−1216 69632
	240	1 20960	1 20960	36 28800	36 28800

The formulas were checked by using simple functions, for instance in the case $n = 9$, the functions $f(t) = t^{10}$ and $f(t) = t^{11}$, with $w = 1$.

The formula used by Eckert, Brouwer, and Clemence is for $n = 9$, and by proper transformations it was carried over into a form that avoided summation. The coefficients obtained were identical with those directly arrived at. Several of the coefficients in the original formula had ten figures in the numerators, while none of those above exceeds eight.

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[1] W. J. Eckert, Dirk Brouwer & G. M. Clemence, "Coordinates of the Five Outer Planets, 1653–2060," *Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac*, v. 12, Washington, 1951.

162.—Historical Note on Root–Finding Machine. A machine which escaped the attention of J. S. Frame[1] is one proposed by Frank T. Freeland.[2] The machine is based upon a method for formation of linkages for the successive powers, published by Freeland in American Journal of Mathematics, v. 3, No. 2.

The machine provides n levers L_k, so connected by linkages that a displacement x of L_1 produces a displacement x^k of L_k, $k = 2$ to n. On each L_k is a pulley P_k with axis parallel to and distance $\frac{1}{2}c_k$ from the axis of L_k, where c_k is the coefficient of x^k. An inextensible cord is passed around the pulleys with the two portions of the cord on P_k parallel, perpendicular to L_k and on the same side of it, using fixed pulleys to change cord direction if necessary. To the end of the cord is fixed a pointer brought parallel to a scale marked off in units of x.

The pointer is set at c_0 with $x = 0$. As L_1 is displaced through values of x, the pointer indicates on the scale the corresponding values of the polynomial. A real root is indicated by zero and the real part of a pair of imaginary roots by a change of direction of the pointer.

Another machine is described for quadratic equations based on a linkage for x^2 suggested by Sylvester. A historical sketch is presented in an appendix, describing machines proposed by Clairaut in 1820 and de Roos in 1879 and mentioning the analytic engine of Charles Babbage. Finally, linkages are described, one for transporting a dimension parallel to itself and one for forming a product based on the differences of squares.

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[1] J. S. Frame, "Machines for solving algebraic equations," *MTAC*, v. 1, 1945, p. 337–353.

[2] Frank T. Freeland, "A machine for the solution of the equation of the n-th Degree," Engineers' Club of Philadelphia, *Proc.*, v. 2, Feb. 23, 1880.

QUERIES—REPLIES

50. Incomplete Hankel Function. (**Q. 43,** v. 8, p. 51).

Put

$$z = (x^2 - 1)^{\frac{1}{2}}, \quad s = y(t^2 - 1)^{\frac{1}{2}}, \quad u = yt = (s^2 + y^2)^{\frac{1}{2}}$$

to obtain

$$\begin{aligned} f(x, y) &= \int_x^\infty (t^2 - 1)^{-\frac{1}{2}} e^{-iyt} dt \\ &= \int_{yz}^\infty u^{-1} e^{-iu} ds \\ &= \int_0^\infty u^{-1} \cos u ds - i \int_0^\infty u^{-1} \sin u ds - \int_0^{yz} u^{-1} ds \\ &+ \int_0^{yz} u^{-1}(1 - \cos u) ds + i \int_0^{yz} u^{-1} \sin u ds. \end{aligned}$$

The first two integrals represent Bessel functions, and the third is an elementary integral. Thus

$$f(x, y) = \{\tfrac{1}{2}\pi Y_0(y) - \sinh^{-1}z + C(y, yz)\} - i\{\tfrac{1}{2}\pi J_0(y) - S(y, yz)\}.$$

Tables[1] of the integrals C and S have been reviewed in RMT 651 (*MTAC*, v. 3, 1948–49, p. 479–482).

A. E.

[1] HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 18, 19: *Tables of Generalized Sine- and Cosine-Integral Functions*, Parts I and II, 1949.

CORRIGENDUM

V. 4, p. 29, l. –13, for xx read 11.